Colorings, monodromy, and impossible triangulations

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Computational geometry in non-euclidean spaces

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Impossible triangulations

Theorem

If a triangulation of $\mathbb{S}^2$ *has exactly two vertices of odd degree, then these are not adjacent.*

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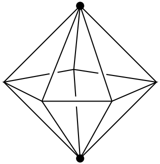
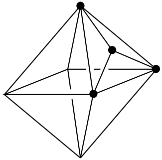
If a triangulation of $\mathbb{S}^2$ has exactly two vertices of odd degree, then these are not adjacent.

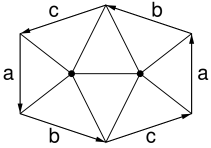
By contrast, one may have:

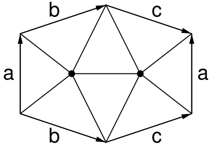
- Two non-adjacent or more than two adjacent odd vertices.





- Two adjacent on the torus and on the projective plane.

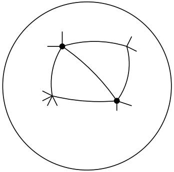




Reduction to even triangulations of polygons

First proof.

Assume such a triangulation exists.



Remove the edge joining the odd vertices (and the adjacent triangles). Get a square, triangulated with all vertices of even degree.

Thus, Theorem ⇔ the square has no even triangulation. □

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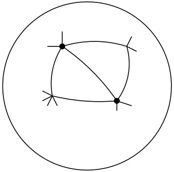

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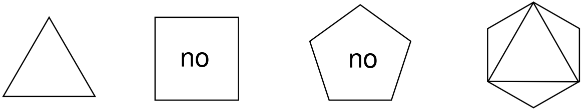
Lemma

An n-gon has a triangulation with all vertices of even degree $\Leftrightarrow$ $n \equiv 0 (\mathrm{mod}\ 3)$.



Even triangulations and colorings

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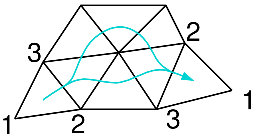
Proof.

An even triangulation can be vertex-colored in 3 colors:
color one triangle; this extends uniquely along any path;
extensions along different paths don't contradict,
due to the even degrees and to the simply-connectedness.



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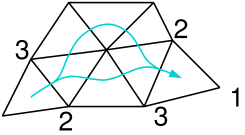

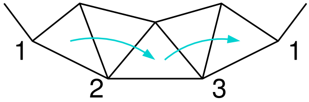

Even degrees $\Rightarrow$ colors of the boundary vertices repeat cyclically $1 \to 2 \to 3 \to 1 \to \cdots$. Hence n is divisible by 3. □

A generalization

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For any $k \in \{2, 3, 4, 5\}$, if degrees of all but two vertices of a triangulation of $\mathbb{S}^2$ are divisible by k, then the exceptional vertices are not adjacent.

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k	2	3	4	5
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But let's introduce a different technique.
Go back to the theorem on two odd vertices.

Color or cover

Given: a triangulated surface (with any number of odd vertices).

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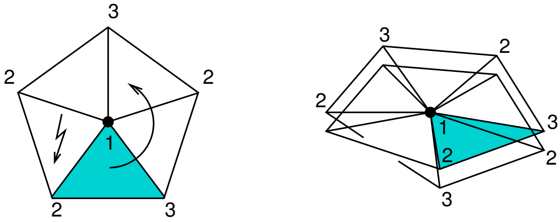

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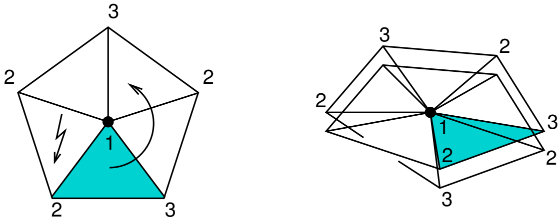

Compare: extending a holomorphic function $f\colon U \to \mathbb{C}$ along different paths can produce different values at the same point. These “branches” of f form the Riemann surface of f.

Coloring monodromy

Definition

Let M be a triangulated surface, Δ_0 a triangle in M, and $a_1, \ldots, a_n$ vertices of odd degree. The coloring monodromy

$$\pi_1(M \setminus \{a_1, \ldots, a_n\}, \Delta_0) \to \mathrm{Sym}(\Delta_0) \cong \mathrm{Sym}_3$$

is a group homomorphism that sends every path starting and ending at Δ_0 to the corresponding vertex re-coloring of Δ_0.

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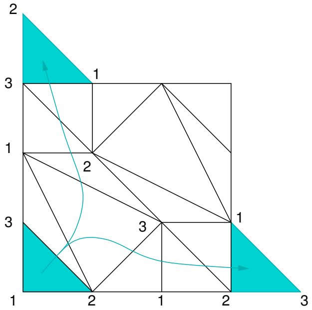
Example

In the 7-vertex triangulation of the torus all vertices have degree 6. The coloring monodromy

$$\mathbb{Z}^2 \cong \pi_1(M) \to \mathrm{Sym}_3$$

permutes the colors in a 3-cycle.



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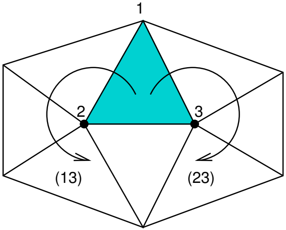
On the other hand, going around a and going around b permutes the colors by two different transpositions.



Hence the image must be the whole Sym_3. Contradiction.

References

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[Joswig’02] Projectivities in simplicial complexes and colorings of simple polytopes.

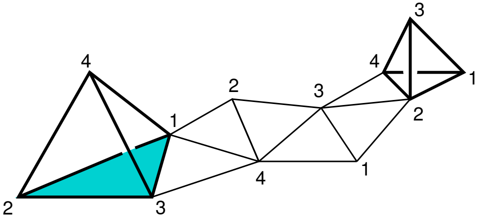
The associated branched cover was introduced and studied in

[I.-Joswig’03] Branched coverings, triangulations, and 3-manifolds.

The focus was on triangulations of $\mathbb{S}^3$ with the edges of odd degrees forming a knot.

Platonic monodromy

For $k = 3, 4, 5$ let P = tetrahedron, octahedron, icosahedron.
Match one of the faces of P with the base triangle of $\mathbb{S}^2$.
Rolling P along a closed path produces a symmetry of P.



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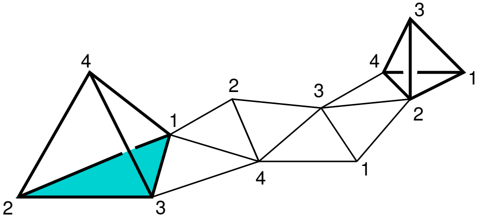

Rolling around a vertex of degree divisible by k produces the identity.
Hence we have the platonic monodromy

$$\pi_1(M \smallsetminus \{a_1, \ldots, a_n\}, \Delta_0) \to \mathrm{Sym}(P)$$

where $a_1, \ldots, a_n$ are vertices of degrees non-divisible by k.

Two vertices of degree $\not\equiv 0 (\mathrm{mod}\ k)$ cannot be adjacent

Assume we have a triangulation of $\mathbb{S}^2$ with only two vertices a, b of degrees non-divisible by k.

The platonic monodromy of this triangulation:

$$\mathbb{Z} \cong \pi_1(\mathbb{S}^2 \setminus \{a, b\}) \to \mathrm{Sym}(P) \qquad (*)$$

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Assume that a, b are adjacent and belong to the base triangle.

Then rolling around a and rolling around b produce two non-commuting symmetries of P.

Hence the image of $(*)$ is non-commutative. Contradiction.

Platonic colorings

Can try to mimic the first proof of the “two odd vertices” theorem.

Instead of platonic monodromy, consider vertex-colorings in 4, 6, or 12 colors, which are the vertices of the tetrahedron, octahedron, icosahedron.

$k = 3$: colorings in 4 colors, where not only adjacent vertices are colored differently, but also those lying “across an edge”.

$k = 4$: colorings in colors $1, 2, \ldots, 6$, where the colors of two vertices across an edge add up to 7 (the dice rule).

$k = 5$: colorings in 12 colors, the coloring rule is complicated...

The minimal colored cover

Holomorphic function $\mapsto$ monodromy $\mapsto$ Riemann surface S with a well-defined function and a branched cover $S \to \mathbb{C} \cup \{\infty\}$

Triangulation $\Sigma \mapsto$ coloring monodromy $\mapsto$ triangulation $\widetilde{\widetilde{\Sigma}}$ that can be colored, together with a branched cover $\widetilde{\widetilde{\Sigma}} \to \Sigma$.

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The minimal colored cover $\widetilde{\widetilde{\Sigma}}$ of a triangulated surface Σ:

$$\{(\Delta, \varphi) \mid \Delta \in \Sigma,\ \varphi\colon \mathrm{Vert}(\Delta) \to \{1, 2, 3\}\} / \sim$$

Two adjacent colored triangles are glued along their common side if their colorings on that side agree.

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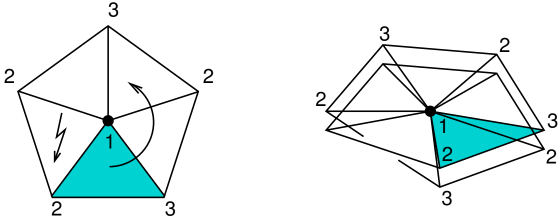

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The *minimal colored cover* $\widetilde{\Sigma}$ *of a triangulated surface* Σ*:*

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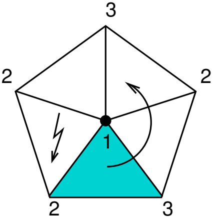

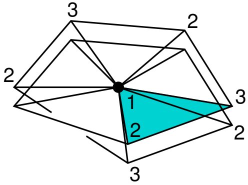

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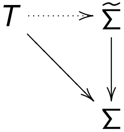
The minimality:
a colored surface that covers Σ covers also $\widetilde{\Sigma}$.



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Given two triangulated surfaces Σ, Σ'.
The space of germs $G(\Sigma, \Sigma')$ consists of triples

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Each triple is a triangle; two triangles are glued side-to-side if they are obtained by "rolling Σ over Σ'".

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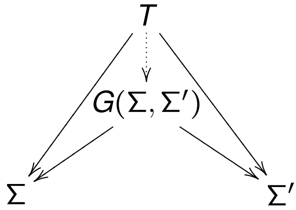
Naturally, $G(\Sigma, \Sigma')$ covers Σ and Σ'.

The universality property:
a surface that covers both Σ and Σ',
covers also $G(\Sigma, \Sigma')$.



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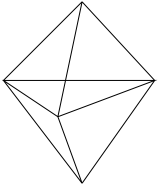
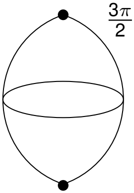
Example





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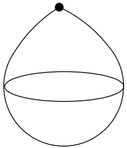
Replace every triangle by a spherical one with the angles $\frac{2\pi}{5}$.
Get a spherical metric with only one cone point (of angle $\frac{12\pi}{5}$).



But there are no spherical cone-metrics with a single cone point.

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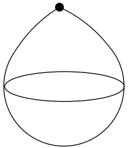

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By contrast, there exist triangulated spheres with 12 vertices of degree 5 and n vertices of degree 6 for all $n \in \{0, 2, 3, 4, \ldots\}$.

The holonomy

Away from the cone points, a cone-surface is locally isometric to the (euclidean plane, sphere, hyperbolic plane).
This allows to develop the neighborhood of every path.

A closed path can develop to a non-closed one.
The neighborhood of the endpoint is “translated and rotated” with respect to the neighborhood of the starting point.

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Definition

Choose a base point $p \in M$ and fix a local isometry of its neighborhood to $(\mathbb{E}^2, \mathbb{S}^2, \mathbb{H}^2)$. The map

$$\pi_1(M \setminus M_{sing}) \to \mathrm{Iso}(\mathbb{E}^2, \mathbb{S}^2, \mathbb{H}^2)$$

is called the holonomy of the cone-surface.

The developing map

If the holonomy is trivial, then the cone-surface can be mapped to $(\mathbb{E}^2, \mathbb{S}^2, \mathbb{H}^2)$ in a locally isometric way.
A cone point of angle $\not\equiv 0(\mathit{mod}\ 2\pi)$ always produces non-trivial holonomy.

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Proposition

If M is simply-connected and the angles around all cone points are multiples of 2π, then the holonomy is trivial, so that there is a map

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In the general case, the developing map goes from the universal cover of $M \setminus M_{sing}$ to $(\mathbb{E}^2, \mathbb{S}^2, \mathbb{H}^2)$.

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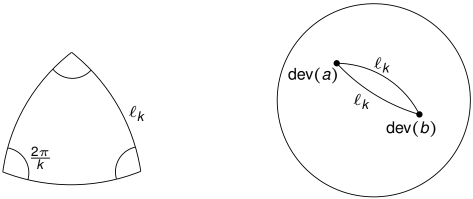
Geometric proof.

Replace each triangle by a spherical one with the angles $\frac{2\pi}{k}$.
Cut along the edge joining the exceptional vertices.
Get a disk with cone points whose angles are multiples of 2π.
Map it to the sphere by the developing map.



The two sides of the slit go to two different geodesics of length ℓ_k with the same endpoints. Contradiction. □

Impossible torus triangulations and non-toral graphs

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New proof: [I., Kusner, Rote, Springborn, Sullivan '13].
Make every triangle equilateral euclidean. Obtain a euclidean metric with two cone-singularities. Study its holonomy.

As a corollary, every graph with these vertex degrees is not embeddable in the torus.