Degrees of freedom in (forced) symmetric frameworks

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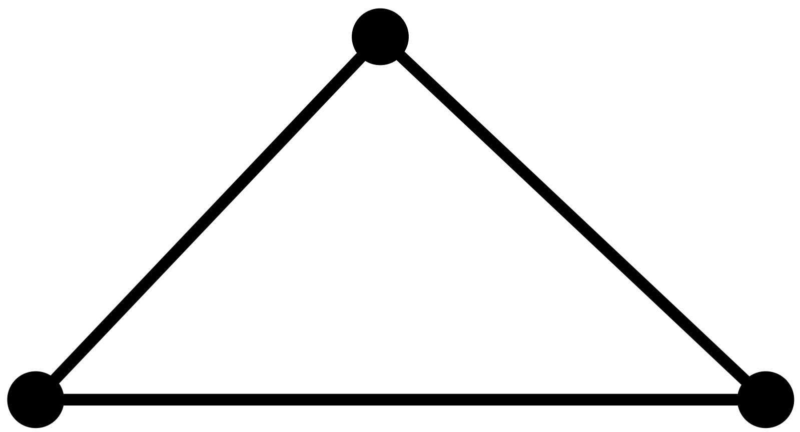
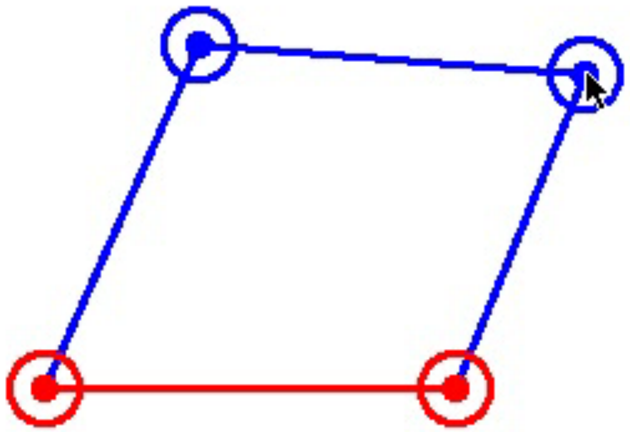
Frameworks





- Graph $G = (V,E)$; *edge lengths* $\ell(ij)$; *ambient dimension d*

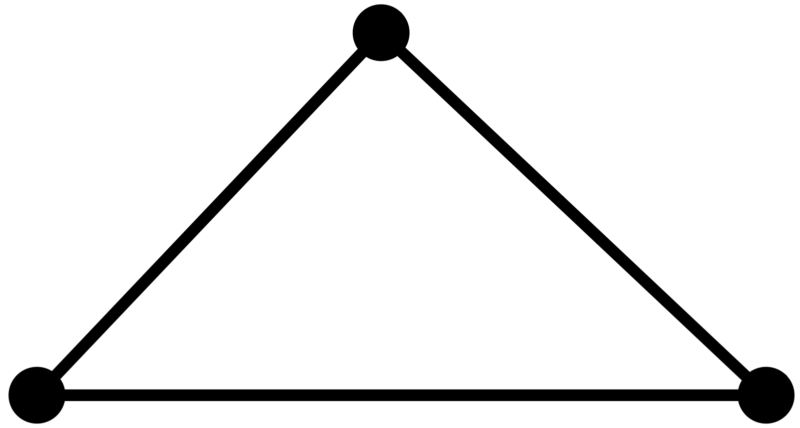
- *Length eqns.*

$$||p_i - p_j||^2 = \ell(ij)^2$$

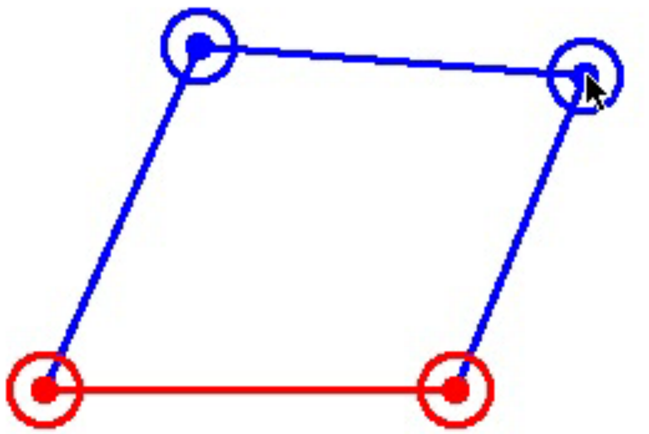
- The p's are a "placement" of *G* / *realization* of (G, ℓ)

Rigidity, flexibility

Rigidity question: which frameworks are rigid?

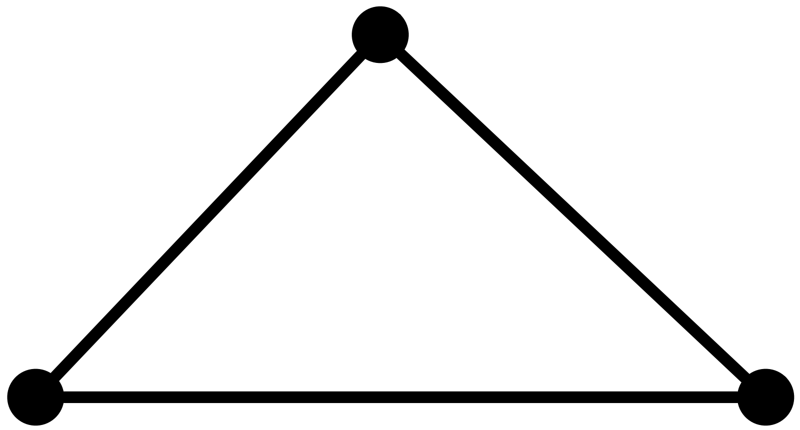


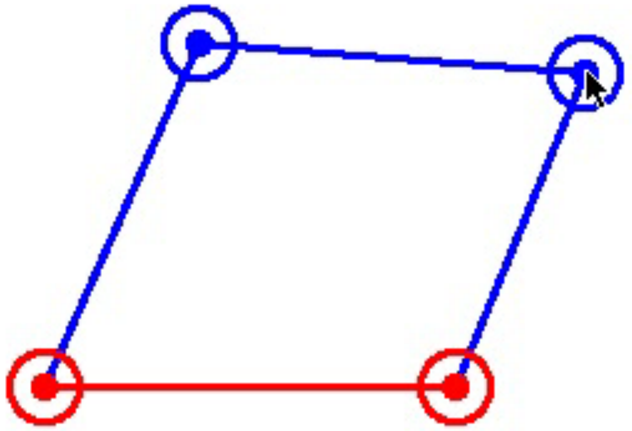
Rigid



Flexible

Frameworks





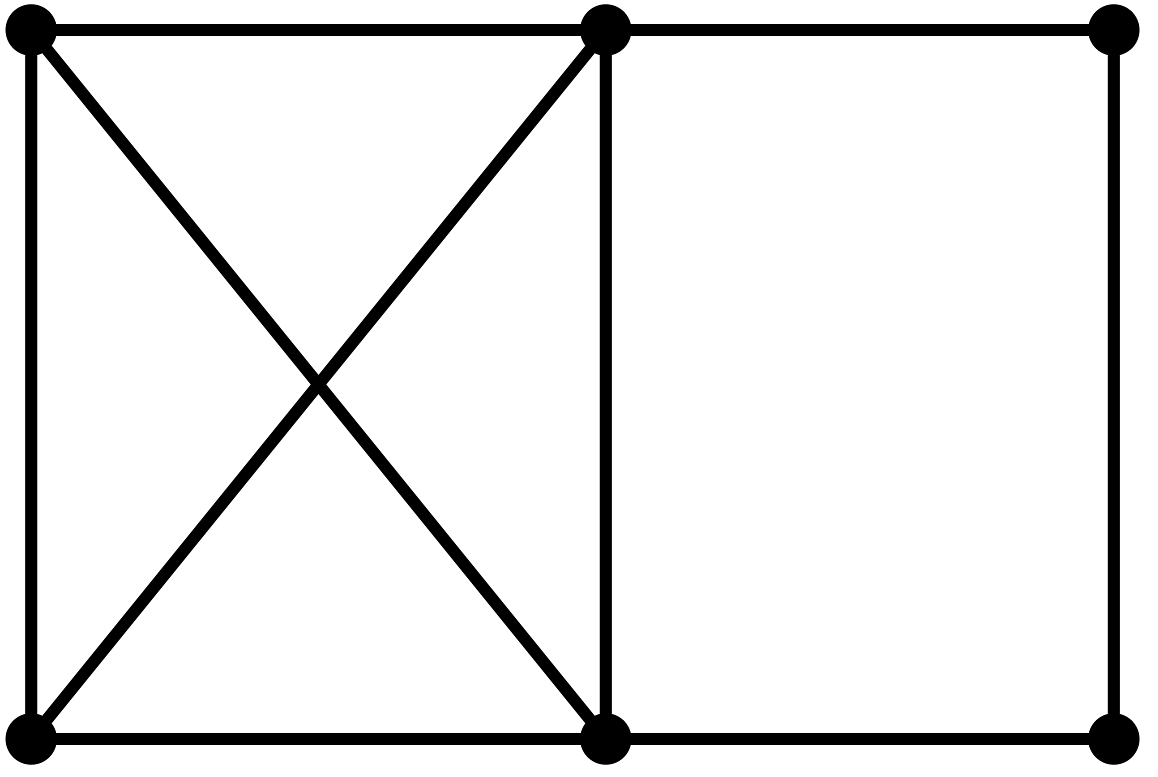
- *Deformation space* = local solutions to

$$||p_i - p_j||^2 = \ell(ij)^2$$

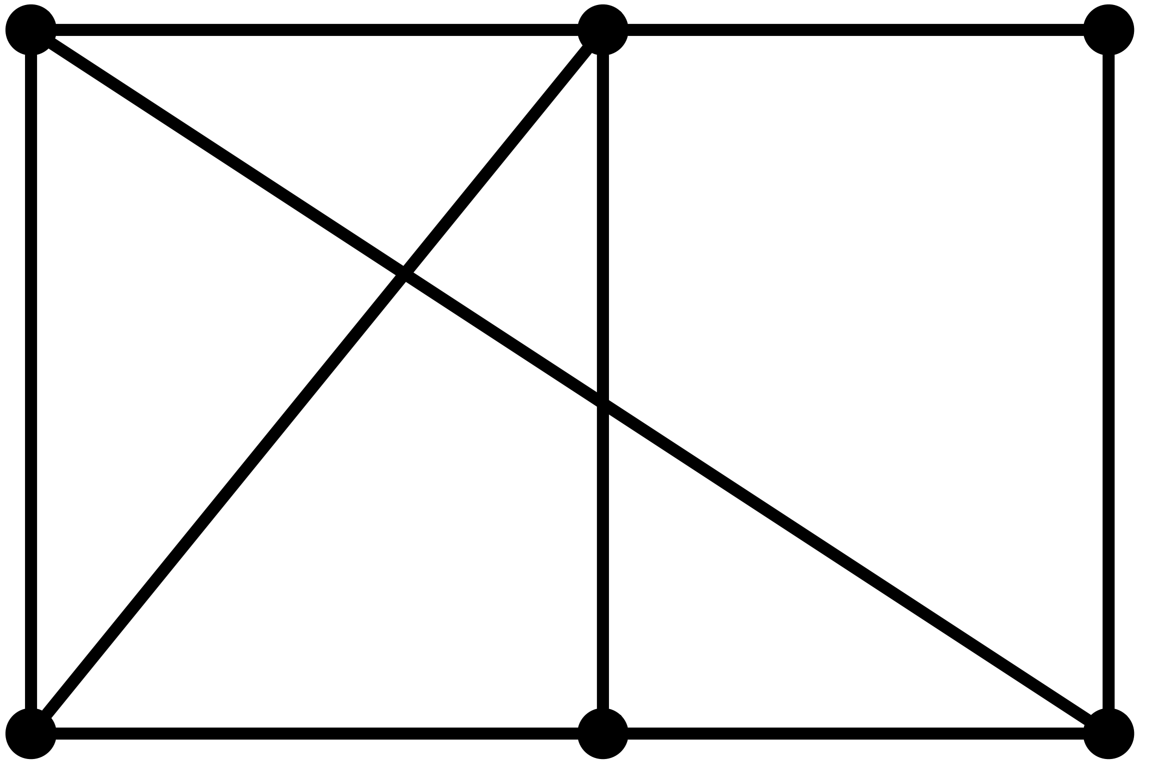
- "mod rigid motions"

- *Degrees of freedom* = dim (deformation space)

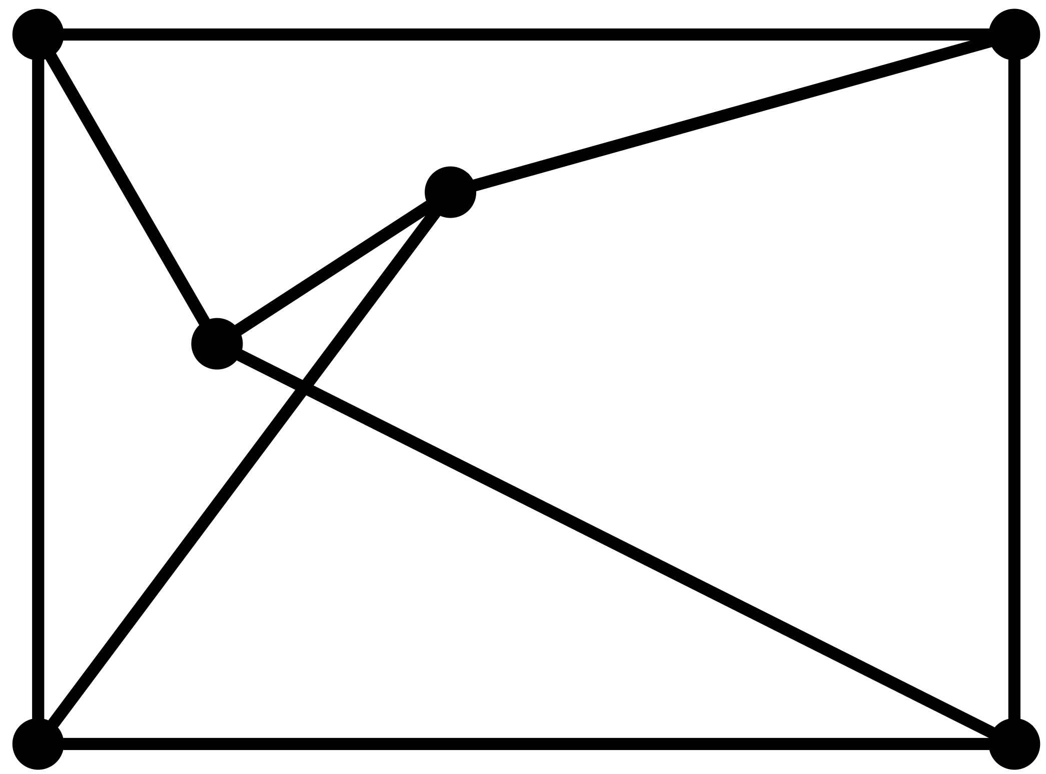
Quiz!



Quiz!

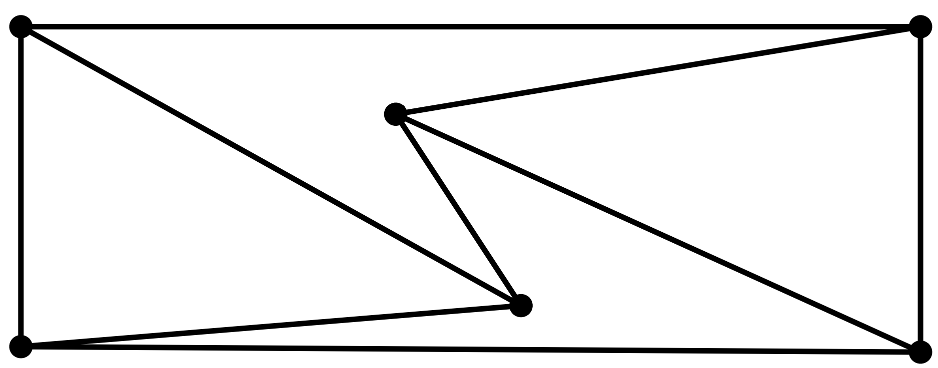


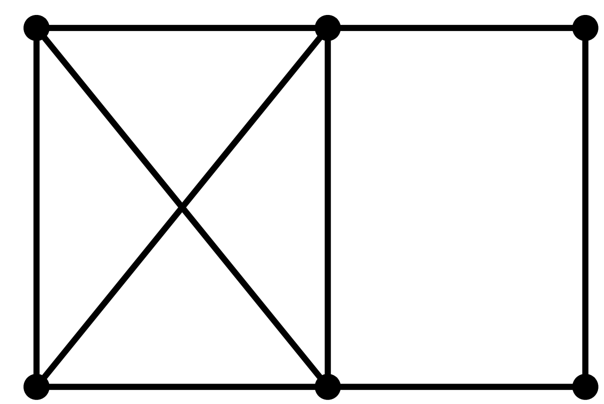
Quiz!



Combinatorial rigidity

Combinatorial rigidity question:
which *graphs* are (*generically*) rigid?





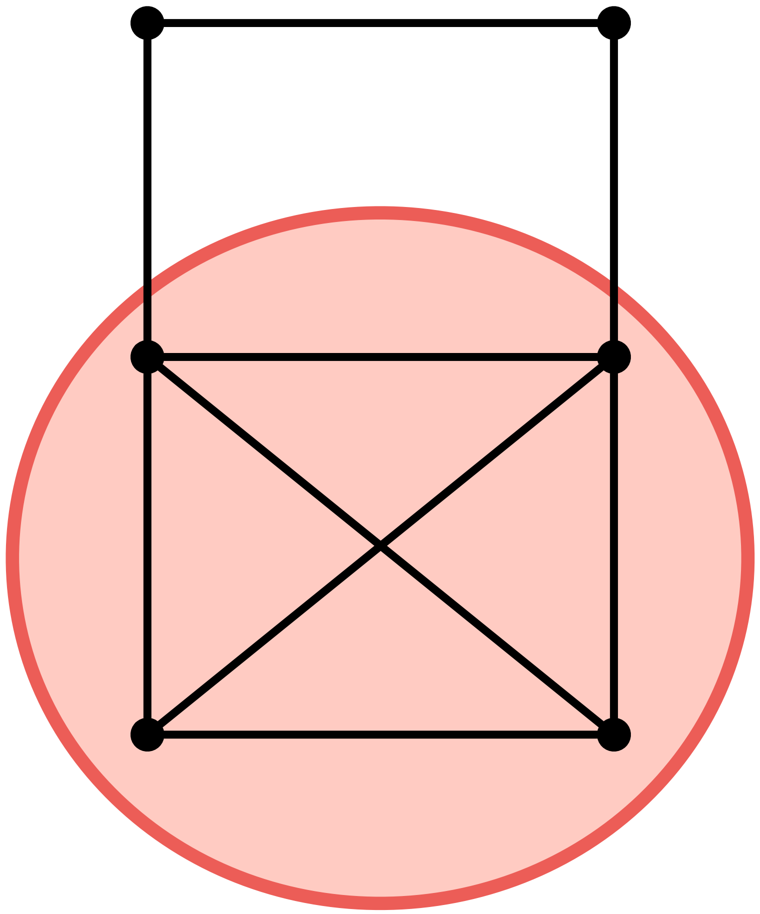
Deformation space is a finite-dimensional algebraic set, well-def'd dimension

Maxwell counting

$$m' \leq 2\, n' - 3$$

eqns. vars. "trivial"

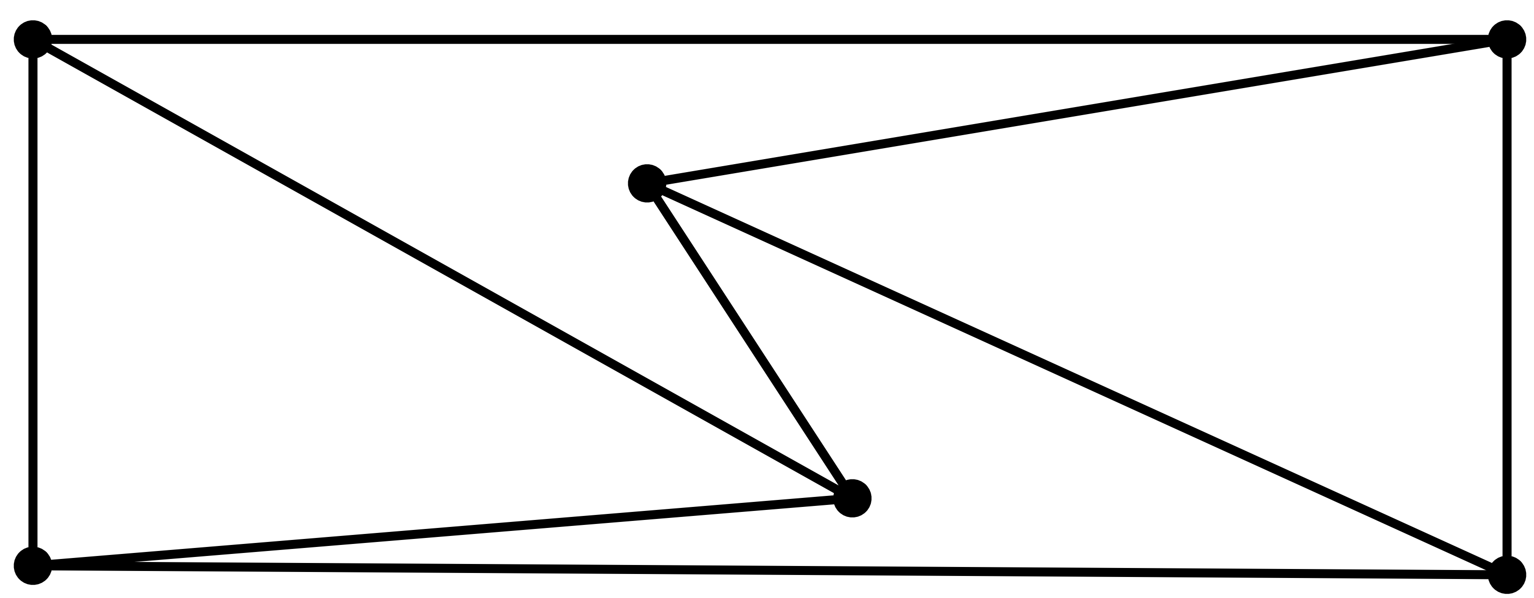
- Each *point* contributes 2 variables
- Each *edge* contributes 1 equation
- Always 3 rigid motions
- *Don't waste any*



Geometry to combinatorics

$$m' \leq 2\, n' - 3$$

- **Theorem** (Laman '70)**:** *Generically*, in $d = 2$, this implies *independence of length equations*. (*Minimal rigidity* if $m = 2n - 3$.)



Why combinatorial rigidity?

- *Generic* frameworks can be general enough
- Can check Laman "$2n - 3$" in $O(n^2)$ time
    - simple "pebble game" algorithms [Hendrickson-Jacobs, Berg-Jordán, Lee-Streinu]
    - no numerical problems
- Useful to know if your problem is non-generic

Open questions

- Which graphs are rigid in $d \geq 3$?

  Naïve Maxwell counting fails.

- Which *infinite and/or symmetric* frameworks are rigid?
  - (Some answers later…)
- What *other geometric constraints* can be analyzed this way?

Genericity vs. universality

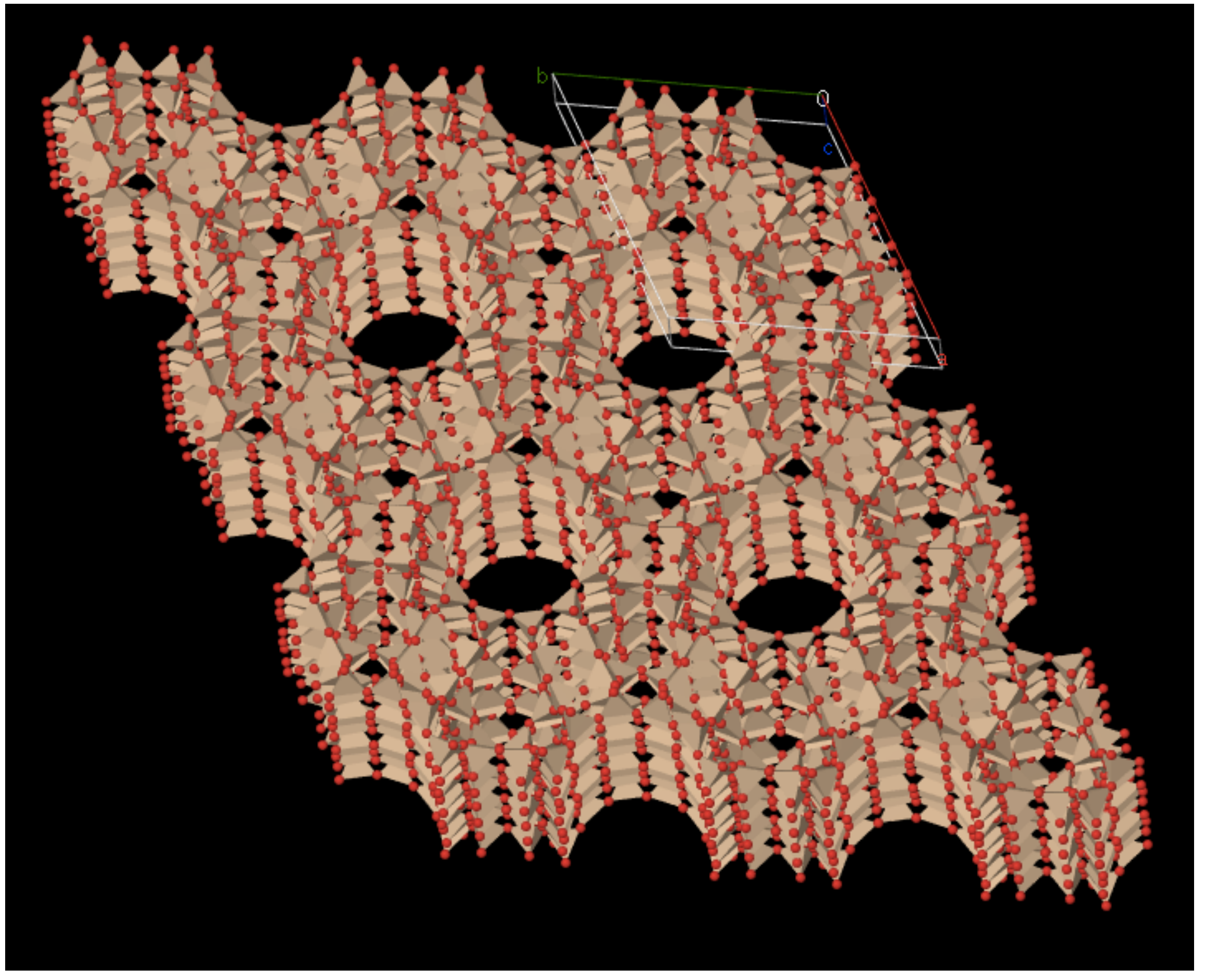
- The rigidity question for *all* frameworks is *universal*

  [Kempe; Mnëv 1988]

  - implies NP-hardness straightforwardly
  - configuration spaces can be homotopy eqv. to any semi-algebraic set
- On the other hand, the hard instances are a *proper algebraic subset* of instances
  - the “non-generic” ones

Application: hypothetical zeolites

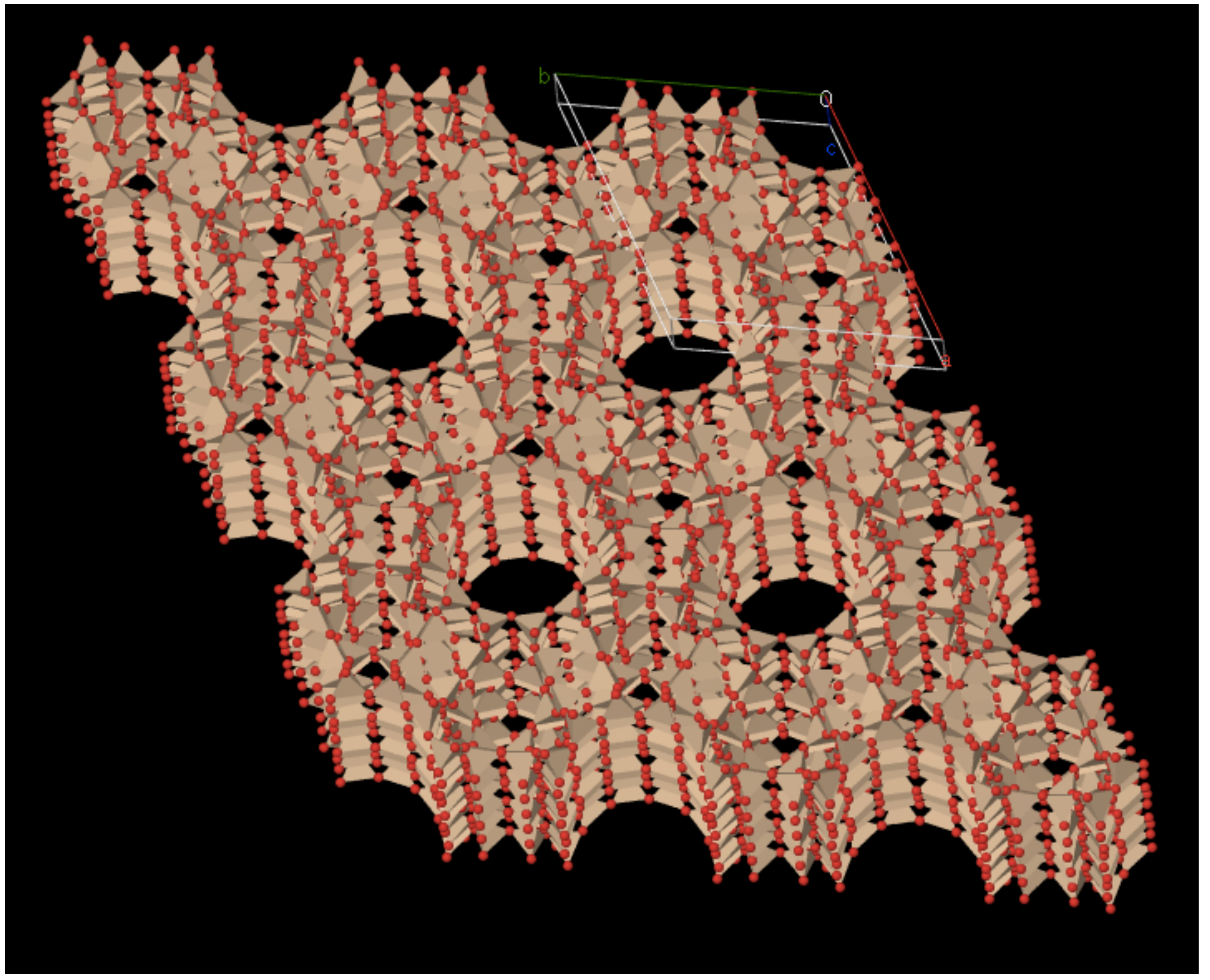
- Aluminosilicates with many industrial applications
- Model as corner-sharing tetrahedra
- Few types known in nature
    - Would like more
- Flexibility is important for function
- Want to know if a combinatorial type is flexible



[from Foster-Treacy]

Application: hypothetical zeolites

- Graph is *infinite*
  - how to compute with it
- Structure is *symmetric*
  - any symmetric structure satisfies *lots* of extra equations
  - *very* non-generic looking
- Want Maxwell-Laman type results



[from Foster-Treacy]

Periodic frameworks

[Borcea-Streinu ‘10]

- A periodic framework (G, ℓ, Γ) is an *infinite* framework with
  - $\Gamma < \mathrm{Aut}(G)$ — Γ free abelian, rank d — finite quotient
  - $\ell(\gamma(ij)) = \ell(ij)$
- A realization $G(p,\Lambda)$ is a realization *periodic* with respect to a *lattice of translations* Λ, which realizes Γ
- Motions *preserve the Γ-symmetry*

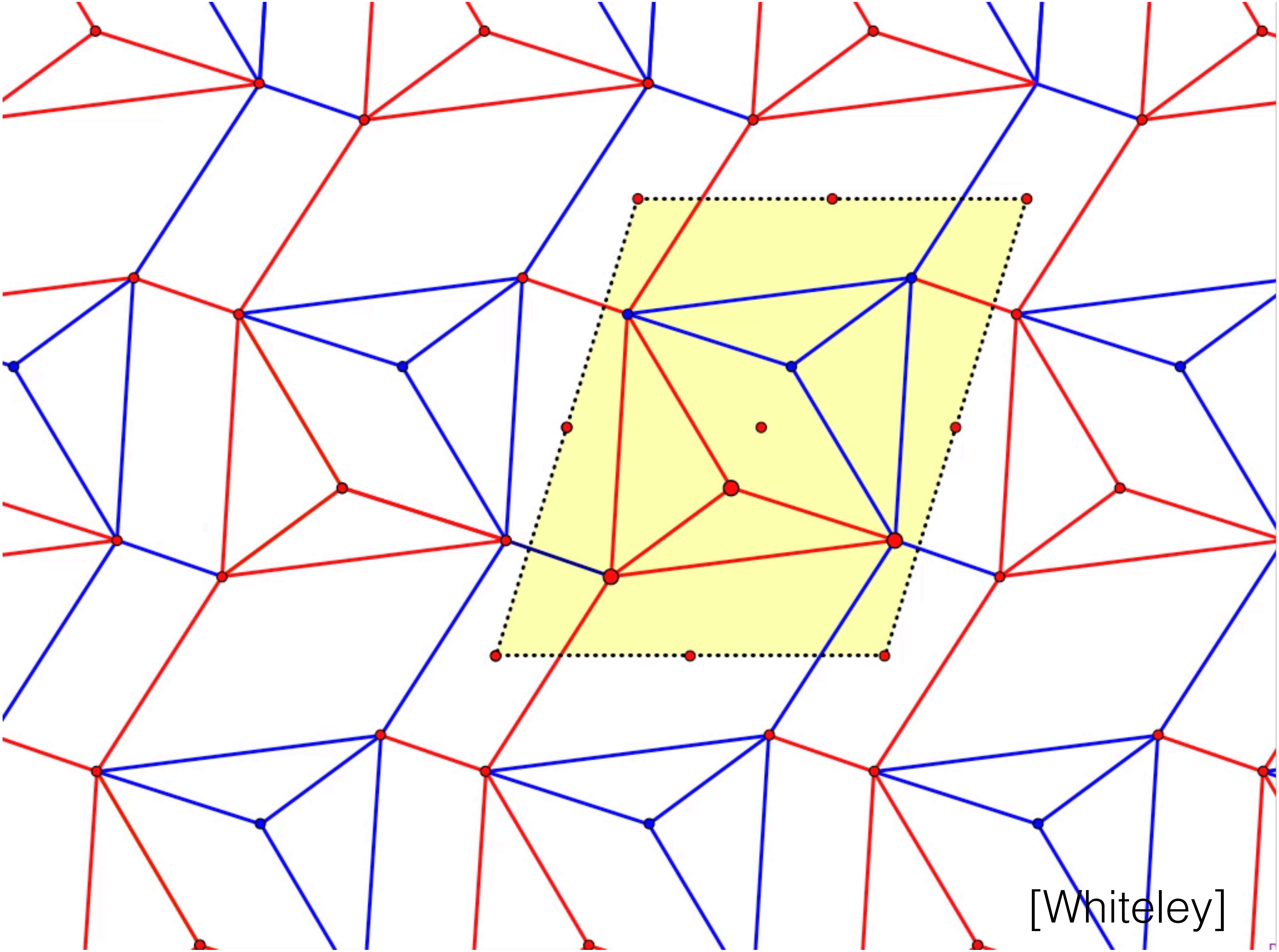
Why periodic frameworks?

[Borcea-Streinu ‘10]

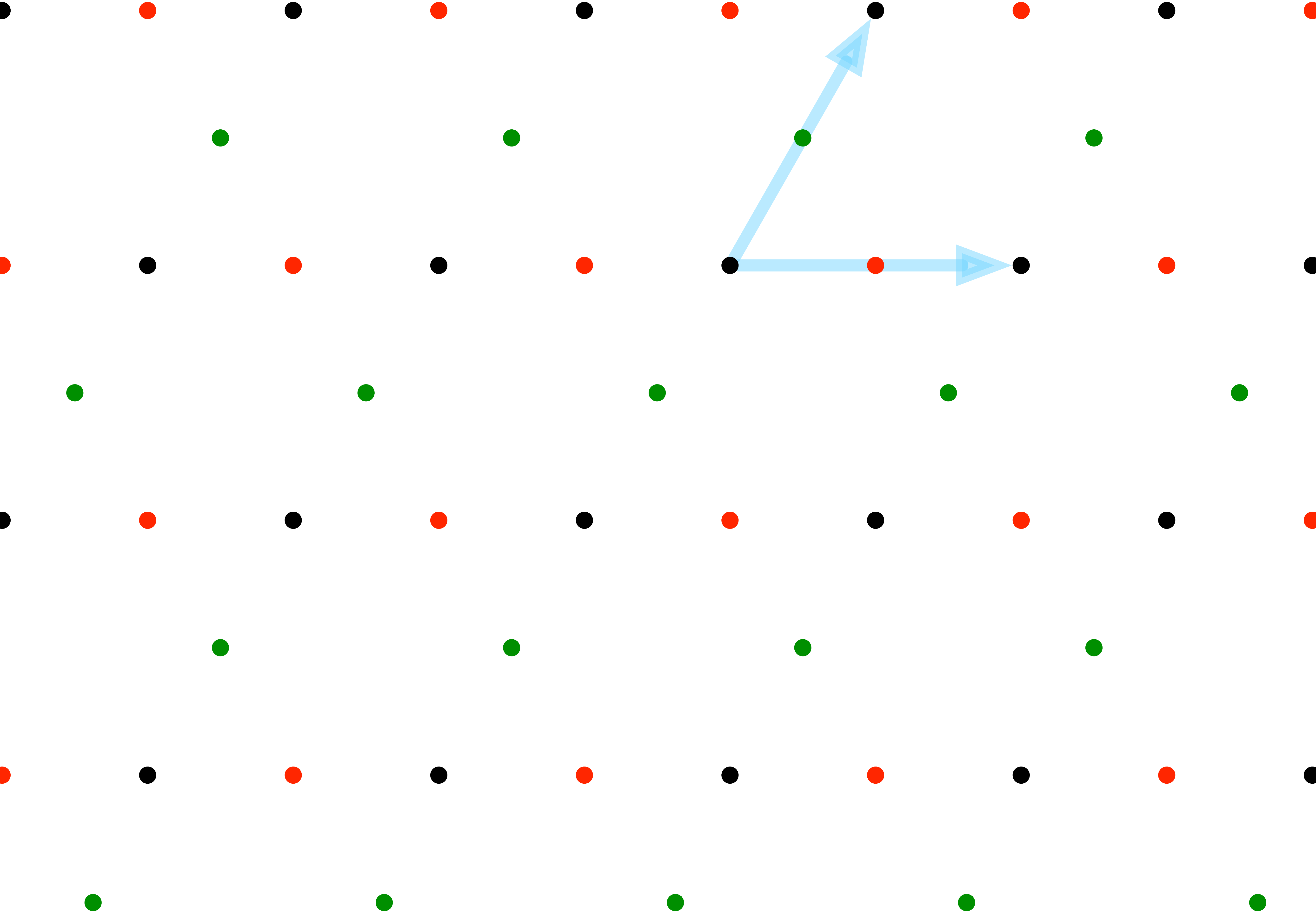
- Can treat configuration spaces with the *same algebraic tools* used for finite frameworks

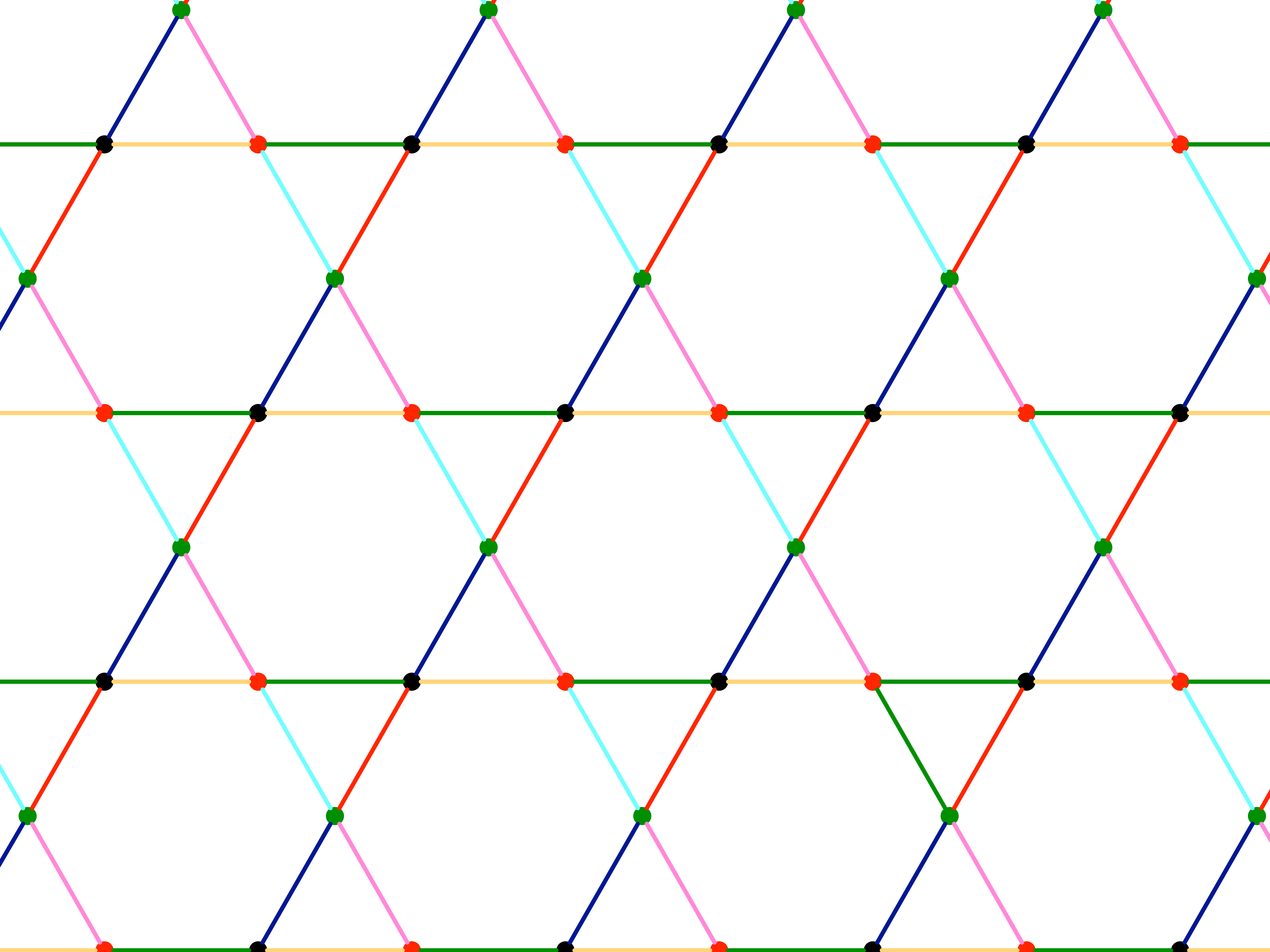
- The *combinatorial type* of a periodic framework is *finite*

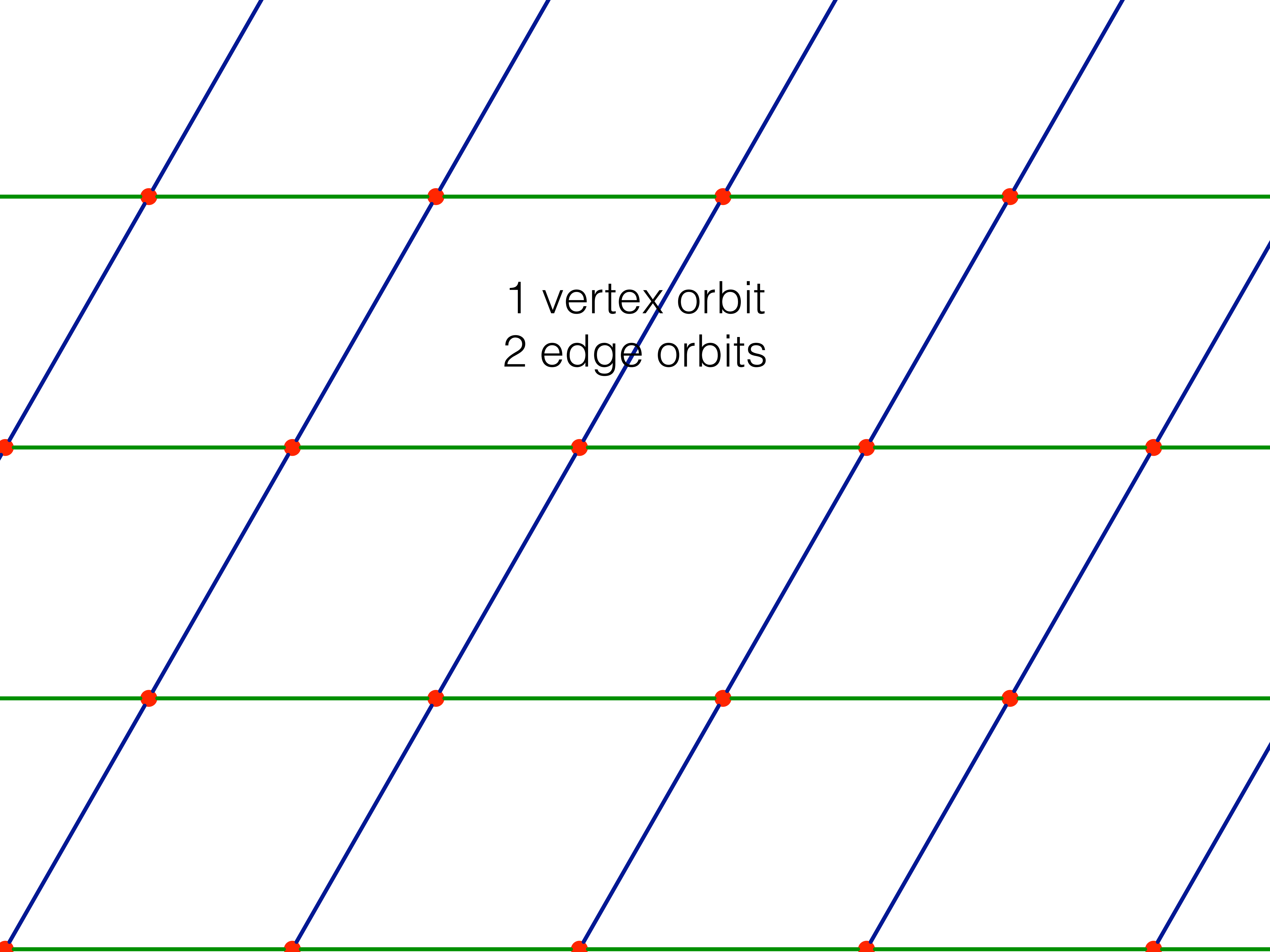
- Preserves duality of static and kinematic infinitesimal rigidity

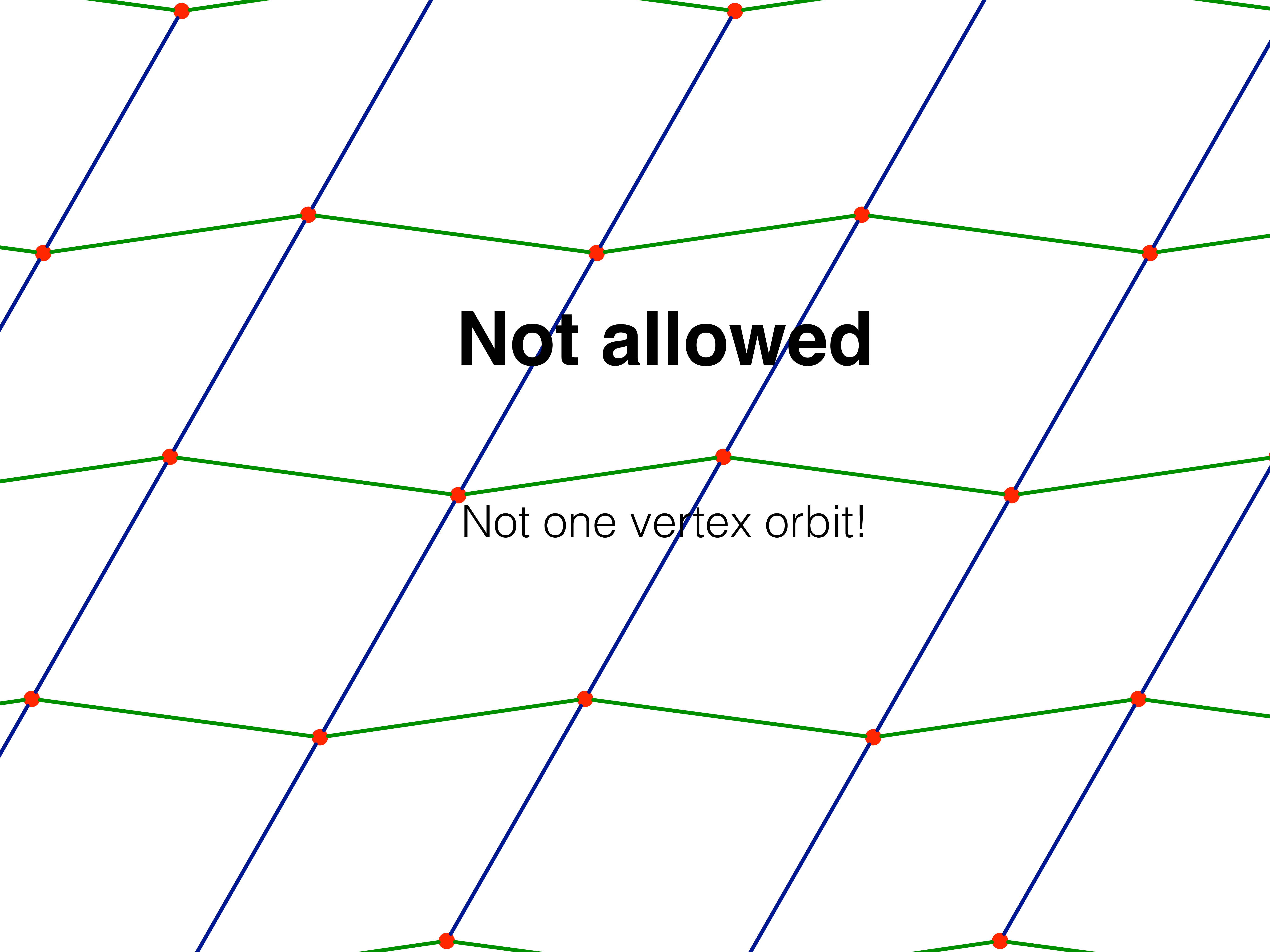


[Whiteley]

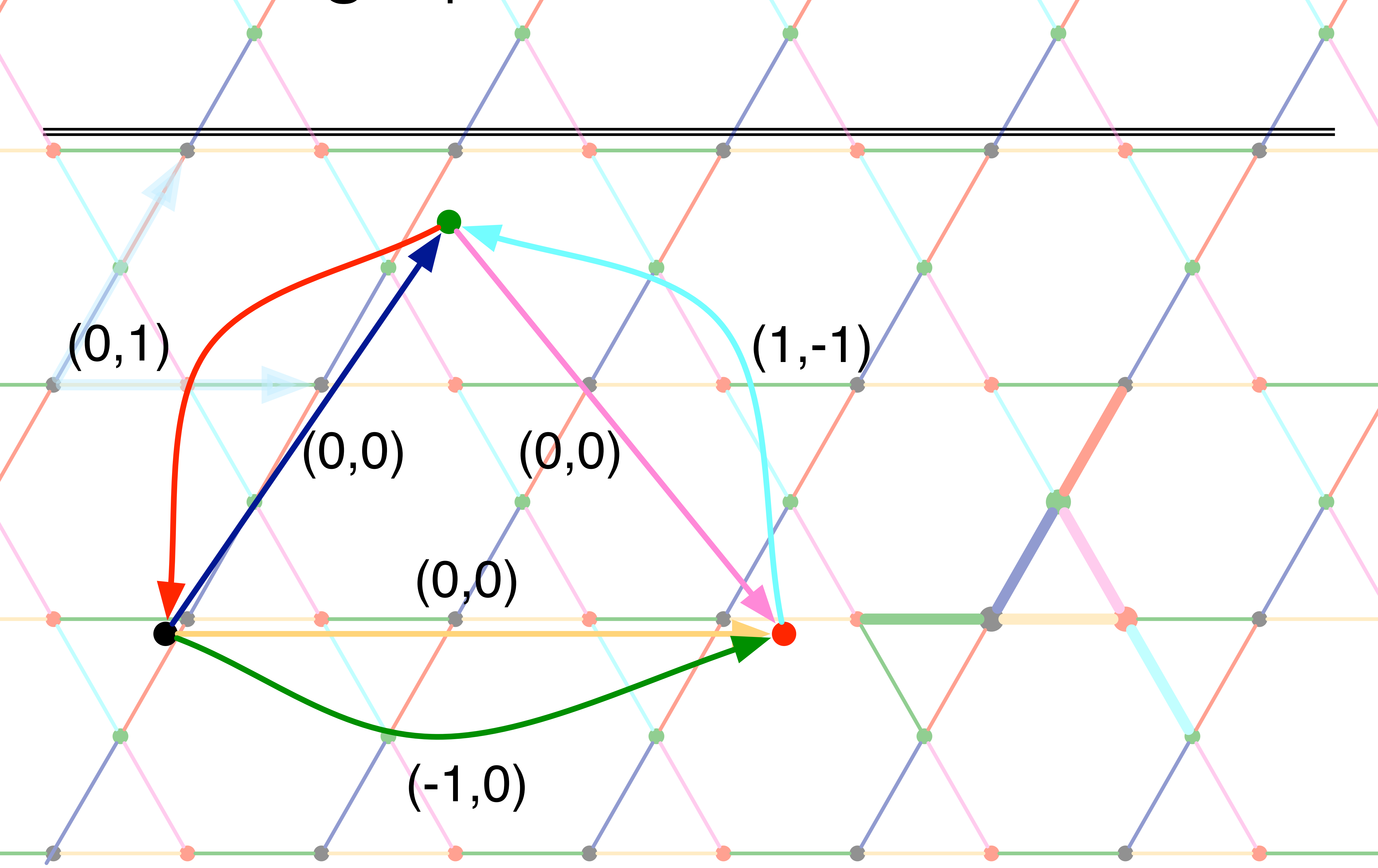





1 vertex orbit
2 edge orbits

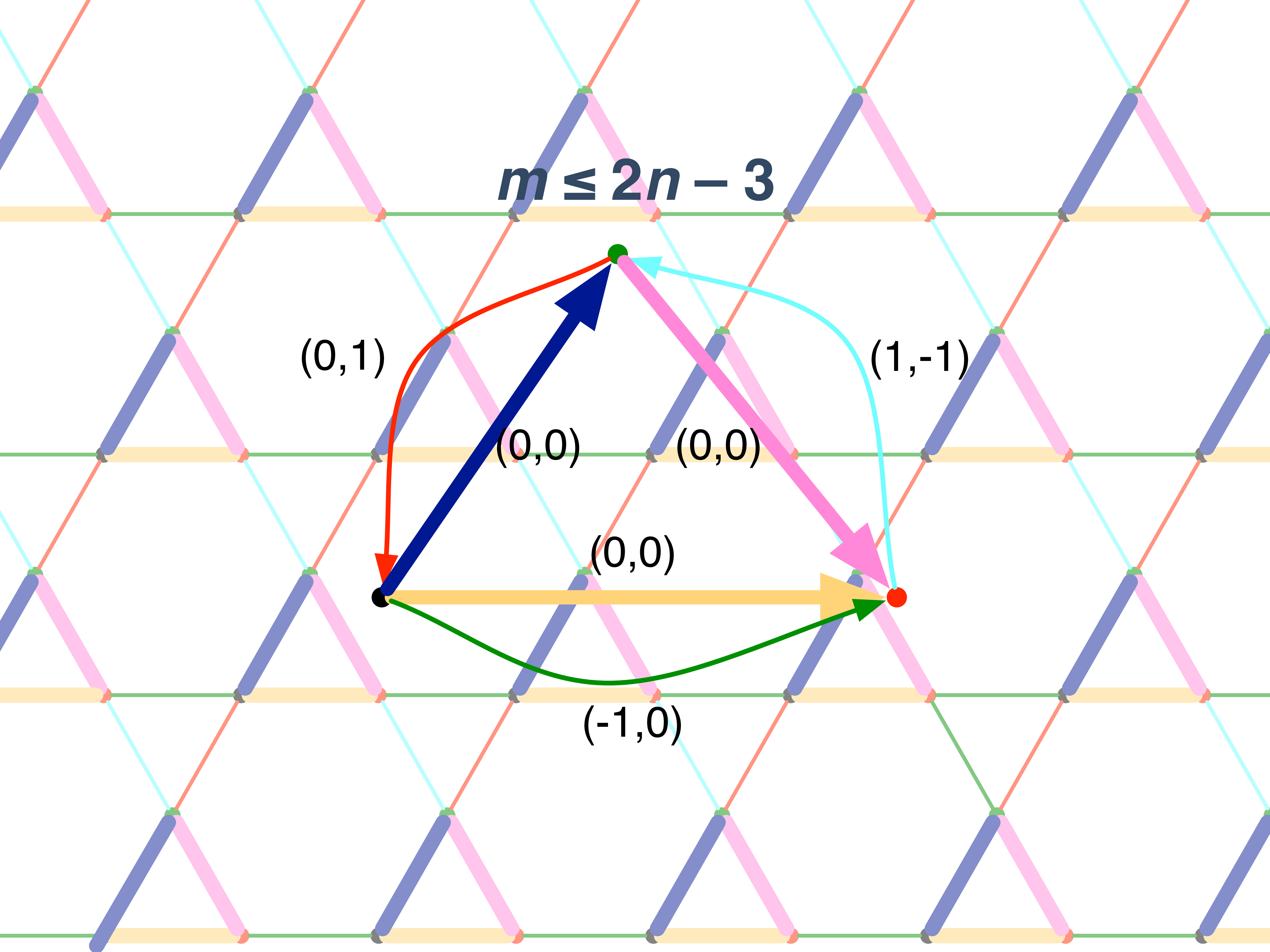

Not allowed
Not one vertex orbit!

Colored graphs

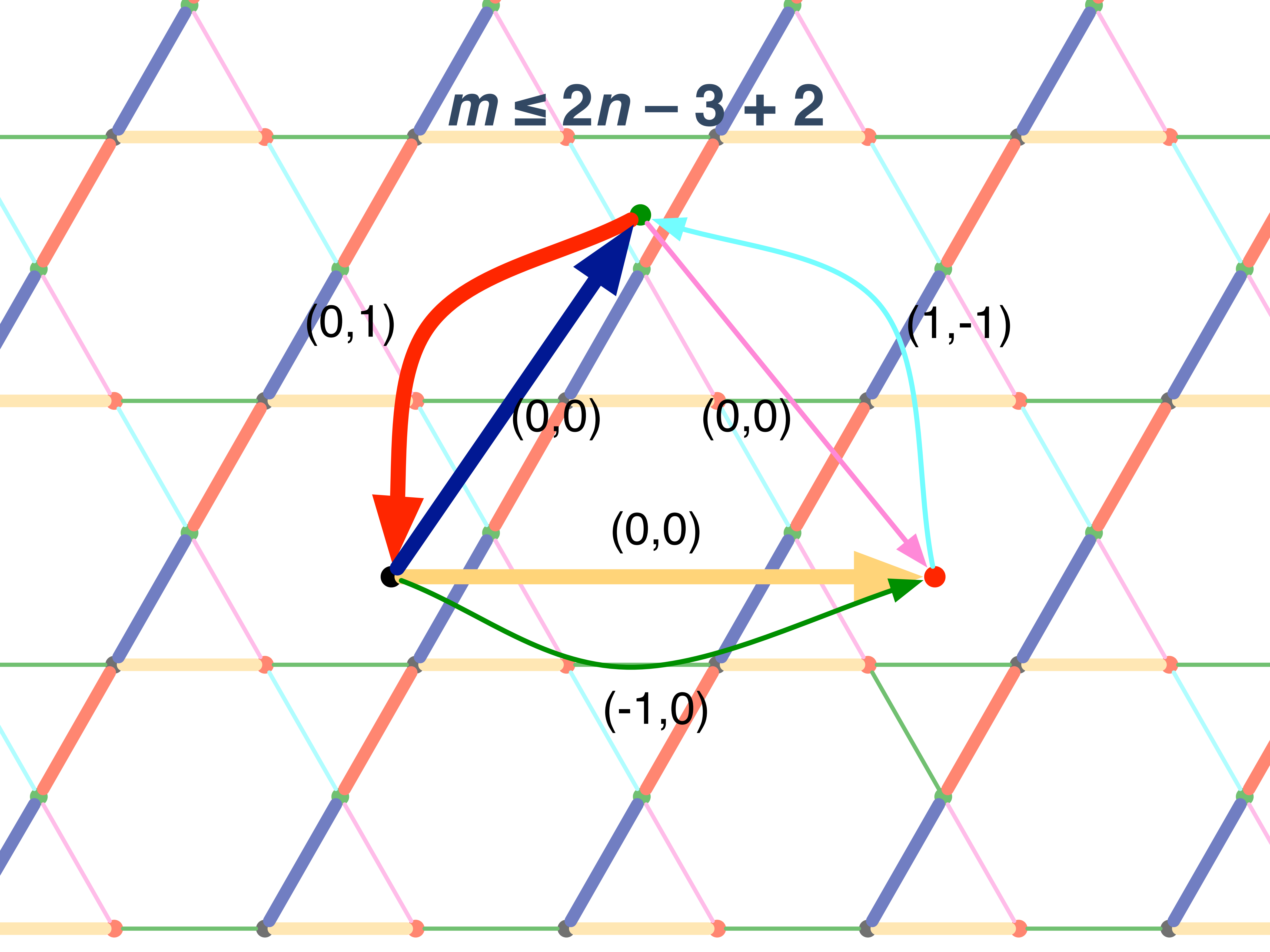


Counting for periodic frameworks

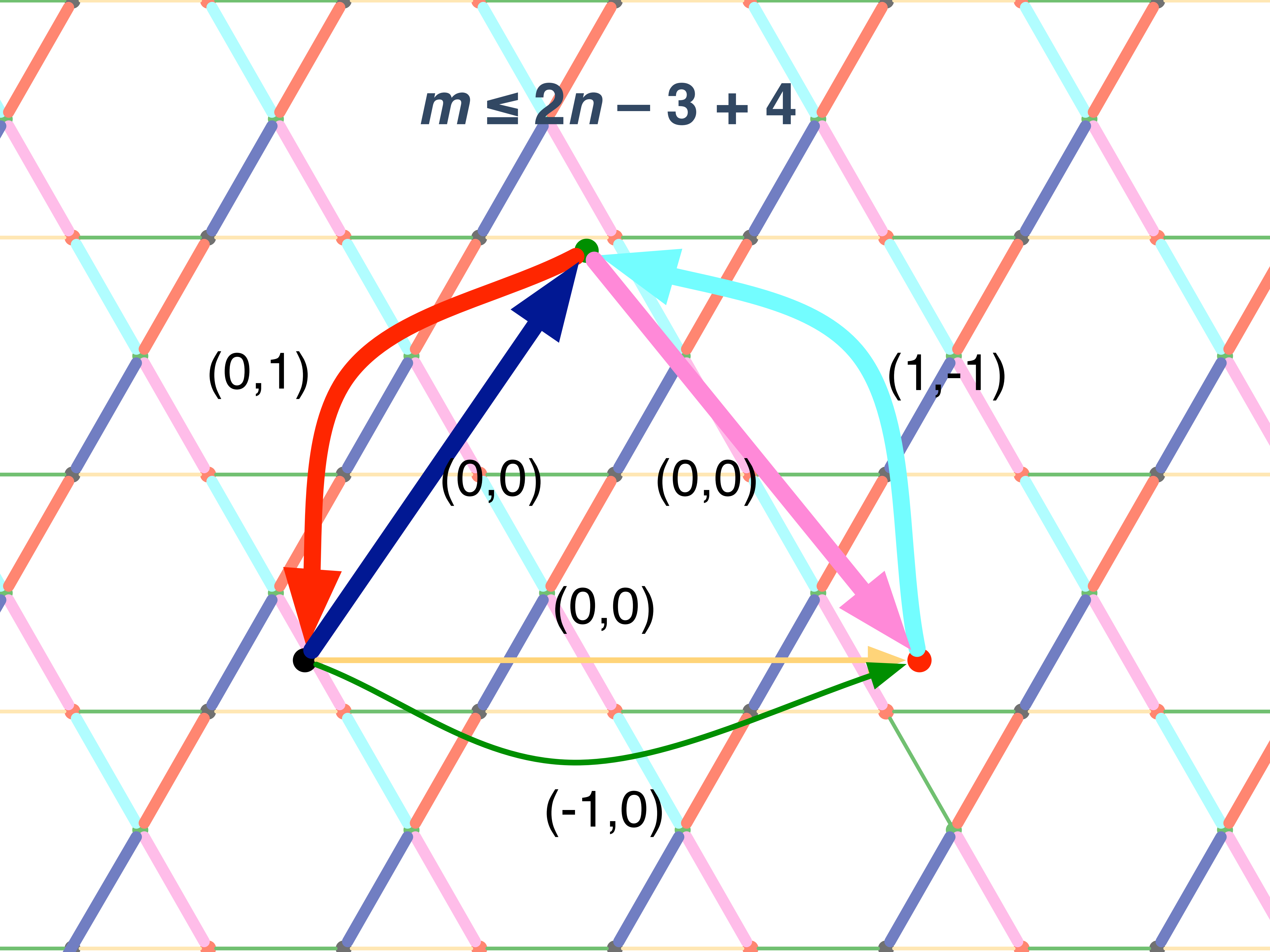
- Each vertex orbit determined by one representative
  - total $2n$ variables from there
- Lattice representation is a 2×2 matrix
  - 4 more variables
- For subgraphs, we will have to distinguish how much of the symmetry group they “see”



$m \leq 2n - 3$
(0,1)
(0,0)
(0,0)
(1,-1)
(0,0)
(-1,0)



m ≤ 2n − 3 + 2
(0,1)
(1,-1)
(0,0)
(0,0)
(0,0)
(-1,0)



$m \leq 2n - 3 + 4$
(0,1)
(0,0)
(0,0)
(1,-1)
(0,0)
(-1,0)

Generic periodic rigidity

- **Theorem** (Malestein-T): For dimension 2

$\mathbf{Z}^2$ rank

connected comps.

$$m' \leq 2(n + k) - 3 - 2(c - 1)$$

characterizes generic independence of length equations.

  - Minimal rigidity if $m = 2n + 1$
  - Generic here is choice of vertex orbits
- Can also check combinatorially

Periodic rigidity variants

- Fixed-lattice (2d) [Ross ’10]
- Fixed-area unit cell (2d) [Malestein-T]
- “Uncolored” quotient graph (*all* d) [Borcea-Streinu ’10]
    - fixed-lattice [Whiteley ’88]

Forced-symmetric counting

- Maxwell heuristic

$$m' \leq 2\,n' - 3$$

eqns. vars. "trivial"

- Symmetry group Γ with representation Φ

$$m' \leq 2\,n' + \mathrm{teich}_{\Gamma}(\Gamma') - \mathrm{cent}_{\Gamma}(\Gamma')$$

eqns. vars. "subgroup flex." "sym.-preserving motions"

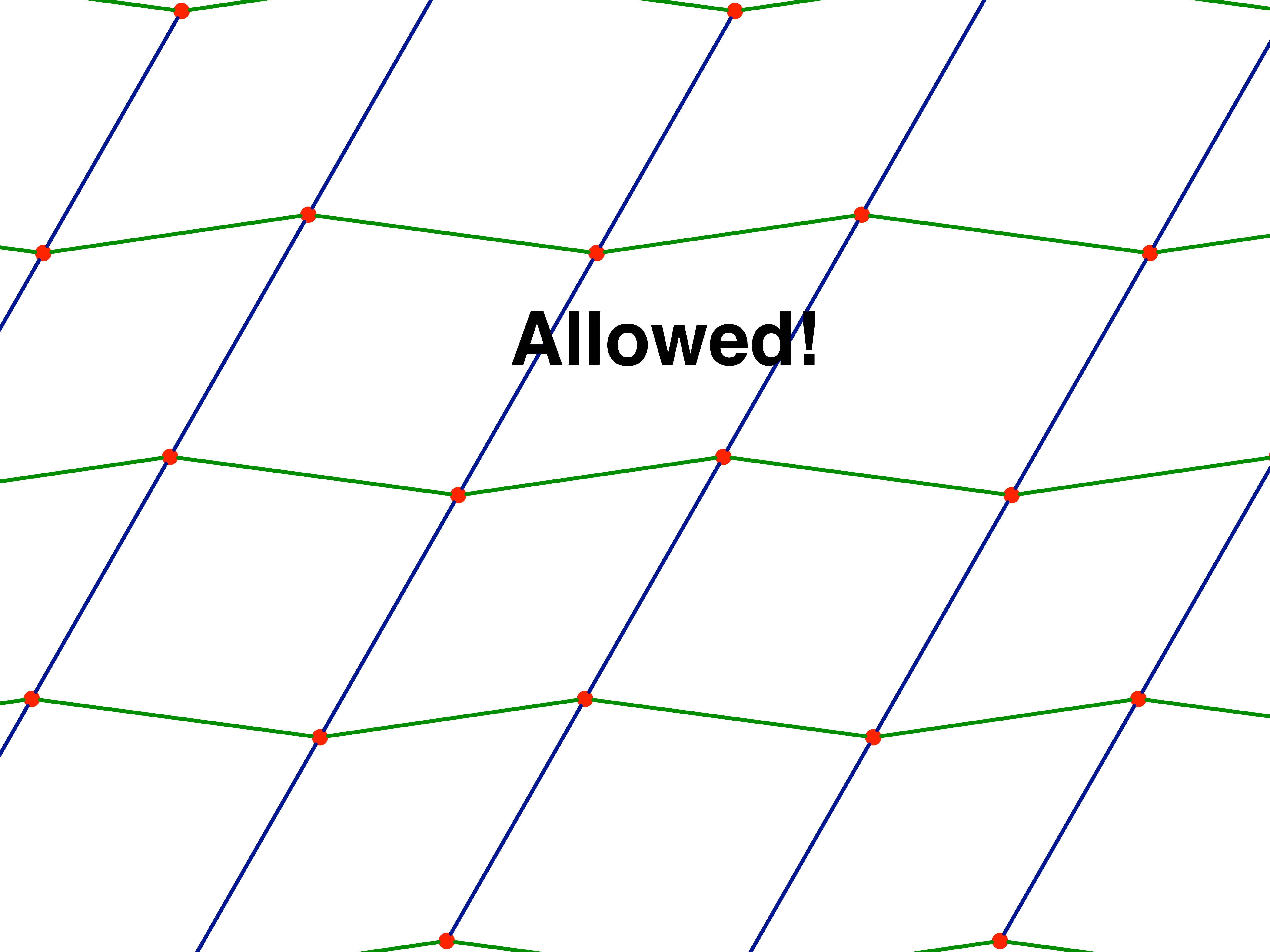
- Γ' is a subgroup associated with a subgraph
- These are all well-defined, depend only on the symmetric lift

Other groups

- Heuristic is sufficient in 2d for
  - Finite order rotations [Malestein-T ’11]
  - One reflection [Malestein-T ‘12]
  - *Odd dihedral groups* [Jordán et al. ‘12]
  - Orientation-preserving wallpaper groups [Malestein-T ‘12]

Further developments

- Periodic body-bar frameworks [Borcea-Streinu-Tanigawa ‘12]
- Forced-symmetric scene analysis [Tanigawa ‘12]
    - Generalizes the families of graphs seen here
    - More groups, more dimensions


Allowed!

Ultrarigidity

[Borcea]

- Let (G, p, L) be a realization of (G,ℓ, Γ)
- (G, p, L) is (periodically) *ultrarigid* if
  - it is rigid
  - for any (finite-index) sub-lattice $\Lambda < \Gamma$, (G, p, L) is a rigid realization of (G, ℓ, Λ)
- Related concept: “ultra 1-d.o.f.” (in 2d)
  - e.g, 4-regular lattices

Why ultrarigidity?

- Naive “infinite frameworks” harder to treat with algebraic-combinatorial ideas [Owen-Power]
- Ultrarigidity is in between infinite frameworks and forced periodicity
- Better hope for combinatorial characterizations

Challenges

- Generic rigidity characterized by the *rank of one matrix* (rigidity/compatibility/… matrix)
  - here there is an infinite family of matrices
- Not completely clear finite ultrarigidity is a generic property
  - Some evidence towards “no”
- We don’t know a priori what failures of ultrarigidity look like

Algebraic characterization

[Connelly-Shen-Smith'14 + Power '13]

- A realization (G, $\mathbf{p}$, $\mathbf{L}$) is infinitesimally ultrarigid if and only if:
  - It is infinitesimally periodically rigid
  - The matrix with *ij*th row, $ij \in E(G,\varphi)$

comp. wise mult

edge direction vector

$$\begin{matrix} & i & & j & \\ (\ldots\ldots & -\mathbf{d}_{ij} & \ldots\ldots\ldots & \mathbf{d}_{ij} \otimes \{\gamma_{ij}^{-1},\omega\} & \ldots\ldots) \end{matrix}$$

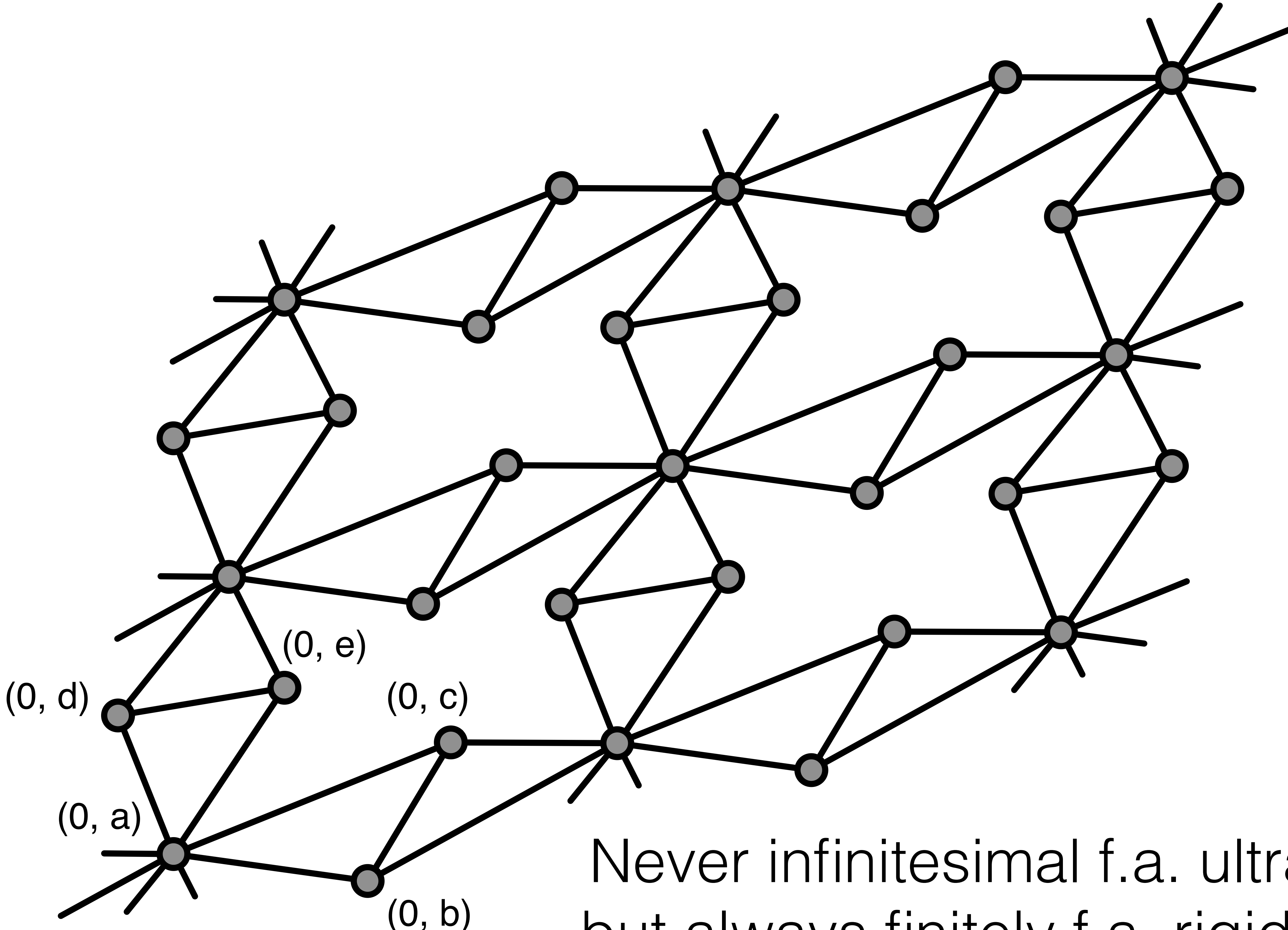
$\{\delta,\omega\} := (\zeta_1^{\delta_1}, \ldots, \zeta_d^{\delta_d})$, ζ_i root of unity

has rank dn for all $\omega \neq \mathbf{1}$

Consequences

- Failures of ultrarigidty fix the lattice
  - [Connelly-Shen-Smith]: Nice geometric argument
  - Direct derivation: Representation theory
- Can check ultrarigidity in finite time [Malestein-T]
  - find a priori bound on order of ζ's

Infinitesimal vs. finite



Never infinitesimal f.a. ultrarigid,
but always finitely f.a. rigid

Counting

- (G,γ) a colored graph with Γ ($\cong Z^d$) colors
- $\psi : \Gamma \longrightarrow \Delta$, epimorphism to a finite cyclic Δ
- "Ultra Maxwell Count" for $(G, \psi(\gamma))$

$$m' \leq d\, n' - d\, T(G, \psi(\gamma))$$

c.c.'s w/ Δ rank > 0

for all ψ.

- *Finitely many* suffice. Sufficient in 2*d* if (G,γ) is *independent* as a periodic framework

Algorithms and combinatorics

- For $m = 2n + 1$, have a combinatorial algorithm polynomial in m (but not γ) for generic infinitesimal periodic ultra rigidity
    - Useful for “small” colors
- For $m = 2n$, have a *polynomial time* algorithm for fixed-area periodic ultrarigidity
    - Via some combinatorial equivalences
- Uses the pebble game, still only $O(n^4)$

Questions

- Finite vs. infinitesimal ultra-rigidity
- “Irrational” points on the RUMS
    - very important in “Mechanical Insulators” theory

      [Kane-Lubensky ‘13]
- Faster algorithms

Kiitos!