Algorithms for embedded graphs

Sergio Cabello
University of Ljubljana
Slovenia

(based on work by/with several people)

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Outline

- Topology and graphs on surfaces
- Algorithmic problems in embedded graphs
- Sample of techniques
A (topological) surface is something that, locally, looks like $\mathbb{R}^2$

We restrict ourselves to compact, orientable surfaces: each is homeomorphic to a sphere with $g$ handles attached to it. We say the genus of the surface is $g$. 
Surfaces – Polygonal schema

A double torus \((g = 2)\) using a polygonal schema
A closed curve is a continuous mapping $\alpha : S^1 \to$ surface

It is simple if it has no self-intersections (injective)
Topological Concepts

- $\alpha, \beta$ closed curves
- $\alpha, \beta$ are homotopic if $\alpha$ can be continuously deformed to $\beta$
- deformation within the surface
Topological Concepts

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Topological Concepts

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Contractible

- $\alpha$ simple closed curve
- $\alpha$ is *contractible* if it is homotopic to a constant mapping

**Theorem:** $\alpha$ contractible and simple $\Rightarrow \alpha$ bounds a disk
Separating

- $\alpha$ closed curve
- $\alpha$ is *separating* if removing its image disconnects the surface
- related to $\mathbb{Z}_2$-homology

**Theorem:** Non-separating $\Rightarrow$ Non-contractible
Embedded Graphs

$G$ is embedded in a surface if:

- each vertex $u \in V(G)$ assigned to a distinct point $u$
- each edge $uv$ assigned to a simple curve connecting $u$ to $v$
- interior of edges disjoint from other edges and $V(G)$
- each face is a topological disk (2-cell embedding)
Embedded Graphs – Polygonal Schema
Representations of Embedded Graphs

- rotation system: for each vertex, the circular ordering of its outgoing edges as DCL.
- coordinate-less DCEL:
  - halfedges
  - vertices
  - faces
  - adjacency relations between them
- flags or gem representation
- ... 

The surface is implicit in the representation of the graph. Surgery should be doable efficiently.
Embeddable vs Embedded

- **planar** graph: can be embedded in the plane
- **plane** graph: a particular embedding
- an embedding can be obtained from the abstract planar graph in linear time
Embeddable vs Embedded

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- **$g$-graph**: can be embedded in $g$-surface
- **embedded** $g$-graph: a particular embedding
- NP-complete: is $G$ a $g$-graph? [Thomassen ’89]
- The problem is fpt wrt genus $g$ [Mohar ’99]
  - “simpler” algorithm by Kawarabayahi, Mohar and Reed 2008
  - $2^{O(g)}n$ time
  - errors in embedding algorithms [Myrvold and Kocay 2011]
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Our scenario

Input: an embedded graph $G$ with (abstract) edge-lengths
Cycles/closed walks in $G$ are closed curves in the surface

Actors: algorithms, topology, and the metric $d_G$
$n \equiv$ complexity of the input graph: $|E(G)|$
The case $g \ll n$ or even $g = O(1)$ is relevant
Algorithmic problems

Input: embedded graph with edge-lengths

- find a shortest non-contractible/non-separating cycle
- find a shortest contractible cycle/walk
- given $\alpha$, find the shortest cycle homotopic/homologous to $\alpha$
- find a cycle shortest in its homotopy/homology class
- max $s-t$ flow
- find a shortest planarizing set
- build a 'good' representation of distances in embedded graphs
- find all replacement paths
- approximate optimum TSP
Shortest non-contractible cycle

- most popular and traditional problem
- subroutine for other problems
  - crossing number: does a graph have crossing number $\leq k$?
  - approximation algorithms for TSP in embedded graphs or near-planar graphs [Demaine, Hajiaghayi, Mohar ’07]
  - numerical analysis for Hodge decomposition
- overlap with analysis of meshes arising from scanned data
  - removal of topological noise [Wood et al. ’04]
  - identification of handles and tunnels [Dey et al. ’08]
Find a shortest non-contractible cycle

- C. Thomassen – $O(n^3 \log n)$  '90
- J. Erickson and S. Har-Peled – $O(n^2 \log n)$  '02
- S. Cabello and B. Mohar – $O(g^{O(g)} n^{3/2} \log n)$  '05
- S. Cabello – $O(g^{O(g)} n^{4/3})$  '06
- M. Kutz – $O(g^{O(g)} n \log n)$  '06
- S. Cabello, E. Colin de Verdiere and F. Lazarus $O(gn^k)$  '12
- S. Cabello, E. Chambers and J. Erickson $O(g^2 n \log n)$  '12

All them also work for non-separating, but no metatheorem.

Directed version, combinatorial bounds, etc.
Shortest contractible curve

- contractible closed walk
  - does not need to be a circuit
  - not difficult to solve in polynomial time
  - $O(n \log n)$ \cite{CabelloDeVosEricksonMohar2010}
  - using \cite{LackiSankowski2011}

- contractible cycle without repeated vertices
  - $O(n^2 \log n)$ \cite{Cabello2010}
  - shortest cycle in planar graph with forbidden pairs

\[ \text{Sergio Cabello} \quad \text{Embedded graphs} \]
Separating cycles

- does it exist any separating cycle without repeated vertices?
  - NP-hard [Cabello, Colin de Verdière, and Lazarus ’10]
  - reduction from Hamiltonian cycle in 3-regular planar graphs
## Summary of some results (up to date?)

<table>
<thead>
<tr>
<th></th>
<th>Cycle</th>
<th>Closed walk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contractible</td>
<td>$O(n^2 \log n)$</td>
<td>$O(n \log n)$</td>
</tr>
<tr>
<td>Separating</td>
<td>NP-hard</td>
<td>???, FPT wrt $g$</td>
</tr>
<tr>
<td>Non-contractible</td>
<td>$O(\min{g^2, n} n \log n)$</td>
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</tr>
<tr>
<td>Tight</td>
<td>$\uparrow$ same</td>
<td>$O(n \log n)$</td>
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<tr>
<td>Splitting</td>
<td>NP-hard</td>
<td>NP-hard, FPT wrt $g$</td>
</tr>
<tr>
<td>Prescribed homotopy</td>
<td>???</td>
<td>nice polynomial</td>
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Unique shortest paths via Isolation Lemma

- unique shortest path between any two vertices
- probabilistically enforced using Isolation Lemma:
  - perturb each edge-length $\ell(e)$ by $k_e \cdot \varepsilon$, where $k_e \in \{1, \ldots, |E|^2\}$ at random
  - each shortest path is unique whp
  - more efficient than lexicographic comparison
- simpler arguments
3-path condition

$P_1, P_2, P_3$ three paths from $x \in V(G)$ to a common endpoint

loops $P_1 + P_3$ and $P_2 + P_3$ contractible

$\Downarrow$

loop $P_1 + P_2$ contractible

- shortest non-contractible loop from $x$ made of two shortest paths
- if $T_x$ shortest path tree from $x$, only loops $\text{loop}(T_x, e)$ are candidates
- there are $|E(G)| - (n - 1)$ candidate loops
3-path condition

Set $L_x$ of loops from $x$ satisfies 3-path condition if:

for any three paths $P_1, P_2, P_3$ from $x$ to a common endpoint, if $P_1 + P_3$ and $P_2 + P_3$ are in $L_x$, then $P_1 + P_2$ is in $L_x$

- $L_x \sim$ zeros in some sense
- contractible loops
- loops with even number of edges
- shortest loop from $x$ outside $L_x$ (non-zero) is made of two shortest paths and an edge
- if membership in $L_x$ is testable in polynomial time, finding shortest loop outside $L_x$ solvable in polynomial time
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- $L_x \sim$ zeros in some sense
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- loops with even number of edges
- shortest loop from $x$ outside $L_x$ (non-zero) is made of two shortest paths and an edge
- if membership in $L_x$ is testable in polynomial time, finding shortest loop outside $L_x$ solvable in polynomial time
- iterate over $x \in V(G)$ for global shortest
Tree-cotree partition - Planar

\( G \) planar. \( T \) a spanning tree
Tree-cotree partition - Planar

$G$ planar. $T$ a spanning tree

$G^* - E(T)^*$ is a spanning tree of the dual graph $G^*$
Tree-cotree partition - General

$G$ embedded graph.
$T$ a spanning tree of $G$
$C \subset E(G)$ cotree: $C^*$ spanning tree of $G^*$ disjoint from $E(T)^*$
$X$ edges not in $T$ or $C$. $X = \{e \in E(G) \mid e \notin E(T) \cup E(C)\}$

- $(T, C, X)$ is a tree-cotree partition
- $X$ has $2g$ edges (orientable) or $g$ edges (non-orientable)
- $(C^*, T^*, X^*)$ a tree-cotree partition of $G^*$
- for any $e \in X$, the cycle in $T + e$ is non-separating
Tree-cotree partition - Example
Tree-cotree partition - Example
Tree-cotree partition - Example
Tree-cotree partition - Cut graph

$G$ embedded graph

$H \subset G$ a cut graph if $G \not\cong H$ is planar

- $(T, C, X)$ is a tree-cotree partition of $G$
- $T \cup X$ is a cut graph: join faces according to $C^*$
- By duality, $C^* \cup X^*$ is a cut graph
Tree-cotree partition - Nice loops

\( G \) embedded graph

\((T, C, X)\) is a tree-cotree partition of \( G \)

\( A = C \cup X \)

\( e \in A \)

\[ \Rightarrow \text{loop}(T, e) \text{ contractible } \iff A^* - e^* \text{ has a tree component} \]

\[ \Rightarrow \text{if loop}(T, e) \text{ contractible } \Rightarrow \text{loop}(T, e) \text{ bounds a disk } D \Rightarrow \]

\[ \text{if } A - e \text{ contains a cotree of } G \cap D \]

\[ \Rightarrow \text{if } A - e \text{ contains a cotree of } G \cap D \Rightarrow \text{deform } e \text{ along } A^* - e^* \]

\[ \Rightarrow \text{cycle homotopic to } A - e \text{ loop}(T, e) \text{ disjoint from } A^* \Rightarrow \]

\[ \text{loop}(T, e) \text{ contractible} \]
Nice loops - Contractible
Nice loops - Contractible
Tree-cotree partition - Nice loops

$G$ embedded graph
$(T, C, X)$ is a tree-cotree partition of $G$
$A = C \cup X$
$e \in A$

$\Rightarrow \text{loop}(T, e)$ separating ifff $A^* - e^*$ disconnected

- $A^* - e^*$ gives a way to merge faces
Embedded graphs
Embedded graphs
Shortest non-contractible loop

$G$ embedded graph
$x \in V(G)$
$L_x$ contractible loops from $x$ Compute shortest loop outside $L_x$
  ▶ compute shortest path tree $T$ from $x$
  ▶ compute dual $A^* = G^* - E(T)^*$
  ▶ compute $B = \{e \in A \mid A^* - e^* \text{ has no tree-component}\}$
  ▶ compute

$$e = \arg \min_{uv \in B} \{d_T(x, u) + d_T(x, v) + |uv|\}$$

  ▶ return loop$(T, e)$$$
⇒$$ linear time per vertex $x$
Theorem

Let $f$ be a specified face in an embedded graph $G$. Preprocess $G$ in $O(g^2 n \log n)$ time such that:

\[
\text{query } (u, v) \in V \times f \quad \xrightarrow{O(\log n) \text{ time}} \quad \text{distance } d_G(u, v)
\]

- compute sp-tree (shortest path) at one vertex
- iteratively move to the neighbor in the face and update the sp-tree
Representation of some distances - Planar

Approach for planar graphs
- compute sp-tree at one vertex of the face
- iteratively move to the neighbor in the face and update the sp-tree
- efficient dynamic data structures to detect what edges come in and out
- reminiscence of kinetic data structures
- use of tree-cotree decomposition
- each (directed) edge appears in a contiguous family of sp-trees (via crossing argument)
- persistence
Parity of crossings of cycles for separating cycles

- \(\alpha\) and \(\beta\) cycles in \(G\)
- \(cr(\alpha, \beta) = \min cr(\alpha', \beta)\) over all tiny deformations \(\alpha'\) of \(\alpha\)
- \(cr_2(\alpha, \beta) = cr(\alpha, \beta) \mod 2\)
- computing \(cr_2(\alpha, \beta)\) is easy
  - invariant under tiny deformations
- useful to work over \(\mathbb{Z}_2\)-homology
- \(\alpha\) separating iff \(cr_2(\alpha, \cdot) = 0\)
- \(cr_2 : H_1 \times H_1\) is well-defined and bilinear
Shortest separating cycle

- \textit{max independent set reduces to: shortest cycle in planar graph with forbidden pairs}

- \textit{surgery to represent the forbidden pairs}

- \textit{separating cycle} \iff \text{crosses any closed curve even nb of times}
Conclusions

- A taste of the algorithmic problems for embedded graphs
- A taste of the techniques
- Gap theory-practice
- Representation-free algorithms
- H-minor-free graphs
- Simple simplicial complexes, like $\beta_i = O(1)$ for all $i$. 

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Embedded graphs