Algorithms for embedded graphs

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(based on work by/with several people)

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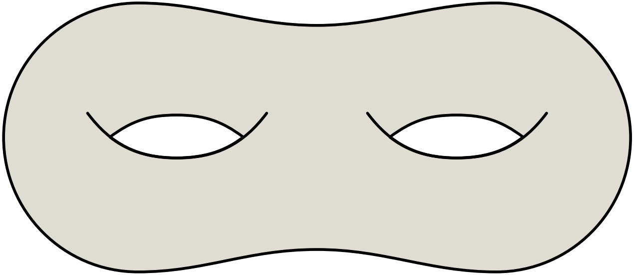
Outline

- Topology and graphs on surfaces
- Algorithmic problems in embedded graphs
- Sample of techniques

Surfaces

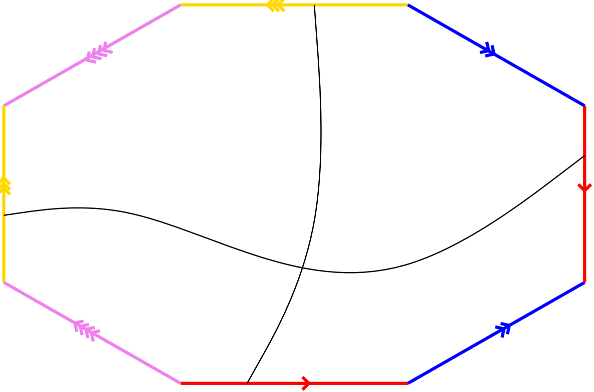
A (topological) surface is something that, locally, looks like $\mathbb{R}^2$



We restrict ourselves to compact, orientable surfaces:
each is homeomorphic to a sphere with g handles attached to it
We say the genus of the surface is g

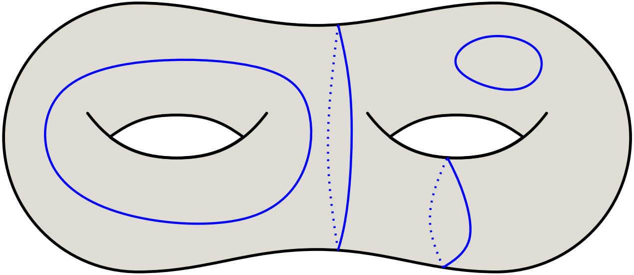
Surfaces – Polygonal schema

A double torus ($g = 2$) using a polygonal schema



Curves on Surfaces

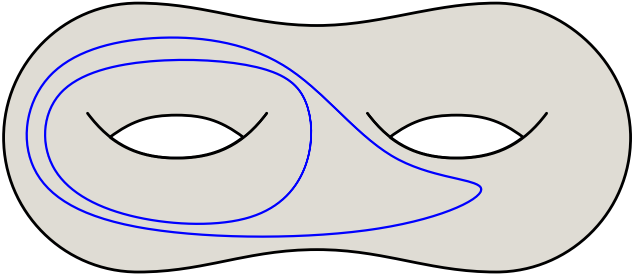
A closed curve is a continuous mapping $\alpha : \mathbb{S}^1 \to$ surface



It is *simple* if it has no self-intersections (injective)

Topological Concepts

- α, β closed curves
- α, β are homotopic if α can be continuously deformed to β
- deformation within the surface



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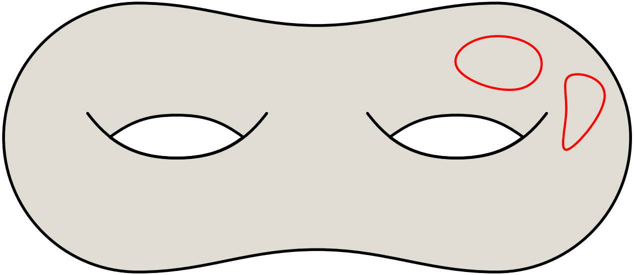

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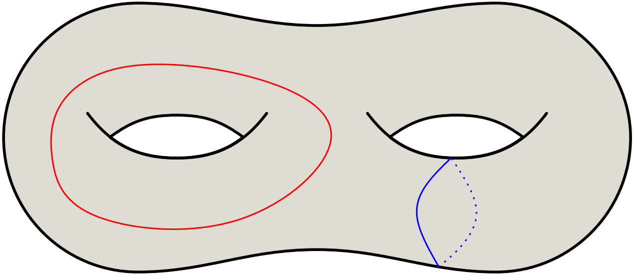
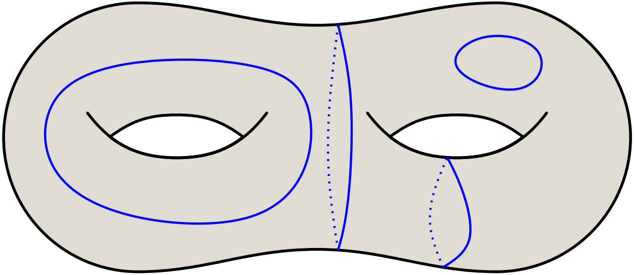


Contractible

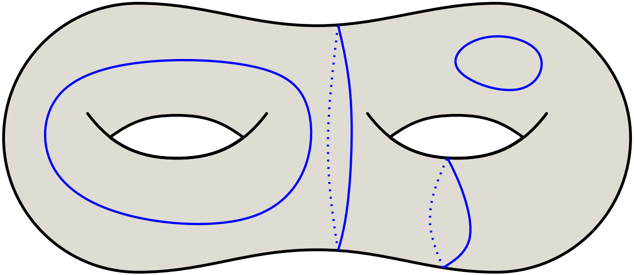
- α simple closed curve
- α is *contractible* if it is homotopic to a constant mapping



Theorem: α contractible and simple $\Rightarrow$ α bounds a disk

Separating

- α closed curve
- α is *separating* if removing its image disconnects the surface
- related to $\mathbb{Z}_2$-homology

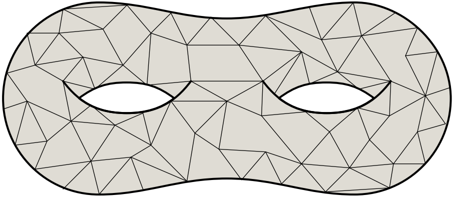


Theorem: Non-separating $\Rightarrow$ Non-contractible

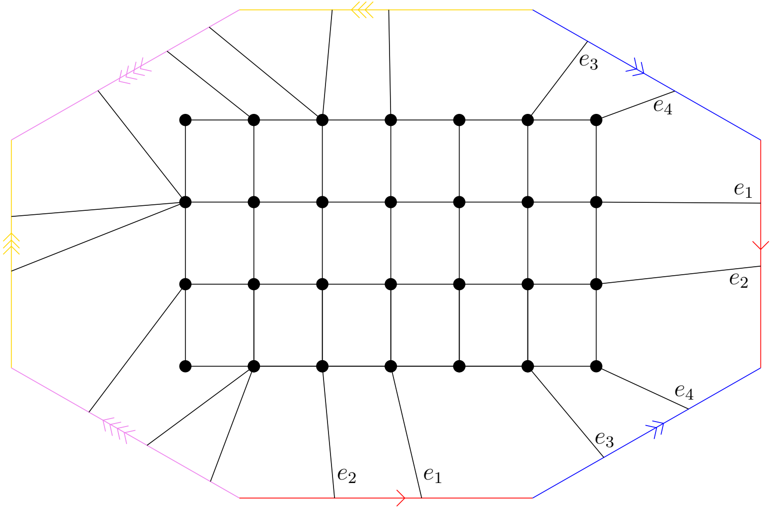
Embedded Graphs

G is embedded in a surface if:

- each vertex $u \in V(G)$ assigned to a distinct point u
- each edge uv assigned to a simple curve connecting u to v
- interior of edges disjoint from other edges and $V(G)$
- each face is a topological disk (2-cell embedding)



Embedded Graphs – Polygonal Schema



Representations of Embedded Graphs

- rotation system: for each vertex, the circular ordering of its outgoing edges as DCL.
- coordinate-less DCEL:
  - halfedges
  - vertices
  - faces
  - adjacency relations between them
- flags or gem representation
- ...

The surface is implicit in the representation of the graph.

Surgery should be doable efficiently.

Embeddable vs Embedded

- planar graph: can be embedded in the plane
- plane graph: a particular embedding
- an embedding can be obtained from the abstract planar graph in linear time

Embeddable vs Embedded

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- g-graph: can be embedded in g-surface
- embedded g-graph: a particular embedding
- NP-complete: is G a g-graph? [Thomassen '89]
- The problem is fpt wrt genus g [Mohar '99]
  - "simpler" algorithm by Kawarabayahi, Mohar and Reed 2008
  - $2^{O(g)}n$ time
  - errors in embedding algorithms [Myrvold and Kocay 2011]

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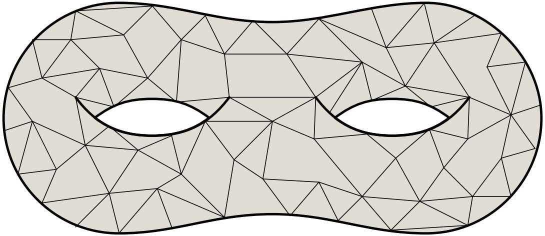
Our scenario

Input: an embedded graph G with (abstract) edge-lengths
Cycles/closed walks in G are closed curves in the surface



Actors: algorithms, topology, and the metric d_G
$n \equiv$ complexity of the input graph: $|E(G)|$
The case $g \ll n$ or even $g = O(1)$ is relevant

Algorithmic problems

Input: embedded graph with edge-lengths

- find a shortest non-contractible/non-separating cycle
- find a shortest contractible cycle/*walk*
- given α, find the shortest cycle homotopic/homologous to α
- find a cycle shortest in its homotopy/homology class
- max *s*-*t* flow
- find a shortest planarizing set
- build a 'good' representation of distances in embedded graphs
- find all replacement paths
- approximate optimum TSP

Shortest non-contractible cycle

- most popular and traditional problem
- subroutine for other problems
  - crossing number: does a graph have crossing number $\leq k$?
  - approximation algorithms for TSP in embedded graphs or near-planar graphs [Demaine, Hajiaghayi, Mohar '07]
  - numerical analysis for Hodge decomposition
- overlap with analysis of meshes arising from scanned data
  - removal of topological noise [Wood et al. '04]
  - identification of handles and tunnels [Dey et al. '08]

Find a shortest non-contractible cycle

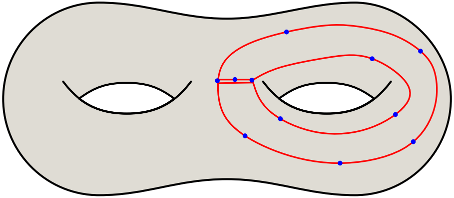
- C. Thomassen – $O(n^3 \log n)$ '90
- J. Erickson and S. Har-Peled – $O(n^2 \log n)$ '02
- S. Cabello and B. Mohar – $O(g^{O(g)} n^{3/2} \log n)$ '05
- S. Cabello – $O(g^{O(g)} n^{4/3})$ '06
- M. Kutz – $O(g^{O(g)} n \log n)$ '06
- S. Cabello, E. Colin de Verdiere and F. Lazarus $O(gnk)$ '12
- S. Cabello, E. Chambers and J. Erickson $O(g^2 n \log n)$ '12

All them also work for non-separating, but no metatheorem.

Directed version, combinatorial bounds, etc.

Shortest contractible curve

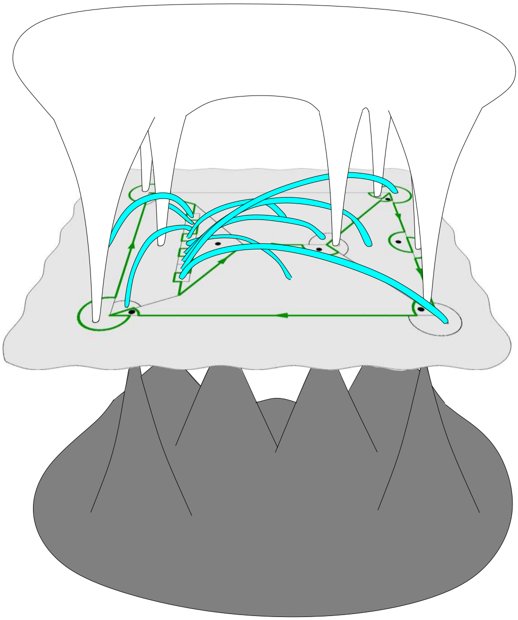
- contractible closed walk
  - does not need to be a circuit
  - not difficult to solve in polynomial time
  - $O(n \log n)$ [Cabello, DeVos, Erickson, Mohar '10] using [Lacki, Sankowski '11]



- contractible cycle without repeated vertices
  - $O(n^2 \log n)$ [Cabello '10]
  - shortest cycle in planar graph with forbidden pairs

Separating cycles

- does it exists *any* separating cycle without repeated vertices?
  - NP-hard [Cabello, Colin de Verdière, and Lazarus '10]
  - reduction from Hamiltonian cycle in 3-regular planar graphs



Summary of some results (up to date?)

	Cycle	Closed walk
Contractible	$O(n^2 \log n)$	$O(n \log n)$
Separating	NP-hard	???, FPT wrt g
Non-contractible	$O(\min\{g^2, n\} n \log n)$	$\leftarrow$ same
Non-separating	$O(\min\{g^2, n\} n \log n)$	$\leftarrow$ same
Tight	$\uparrow$ same	$O(n \log n)$
Splitting	NP-hard	NP-hard, FPT wrt g
Prescribed homotopy	???	nice polynomial
Prescribed homology	NP-hard, FPT wrt g	$\leftarrow$ same

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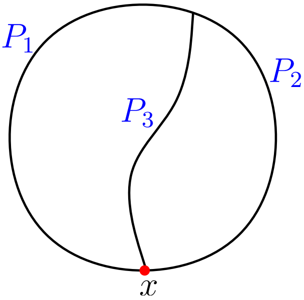
Unique shortest paths via Isolation Lemma

- unique shortest path between any two vertices
- probabilistically enforced using Isolation Lemma:
  - perturb each edge-length $\ell(e)$ by $k_e \cdot \varepsilon$, where $k_e \in \{1, \dots, |E|^2\}$ at random
  - each shortest path is unique whp
  - more efficient than lexicographic comparison
- simpler arguments

3-path condition

P_1, P_2, P_3 three paths from $x \in V(G)$ to a common endpoint

loops $P_1 + P_3$ and $P_2 + P_3$ contractible
$\Downarrow$
loop $P_1 + P_2$ contractible



- shortest non-contractible loop from x made of two shortest paths
- if T_x shortest path tree from x, only loops $loop(T_x, e)$ are candidates
- there are $|E(G)| - (n-1)$ candidate loops

3-path condition

Set L_x of loops from x satisfies 3-path condition if:

for any three paths P_1, P_2, P_3 from x to a common endpoint, if $P_1 + P_3$ and $P_2 + P_3$ are in L_x, then $P_1 + P_2$ is in L_x

- $L_x \sim$ zeros in some sense
- contractible loops
- loops with even number of edges
- shortest loop from x outside L_x (non-zero) is made of two shortest paths and an edge
- if membership in L_x is testable in polynomial time, finding shortest loop outside L_x solvable in polynomial time

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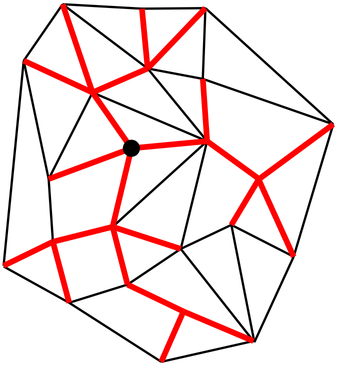
- if membership in L_x is testable in polynomial time, finding shortest loop outside L_x solvable in polynomial time
- iterate over $x \in V(G)$ for global shortest

Tree-cotree partition - Planar

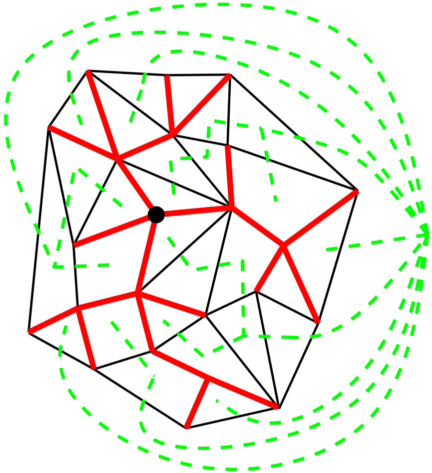
G planar. T a spanning tree



Tree-cotree partition - Planar

G planar. T a spanning tree

$G^* - E(T)^*$ is a spanning tree of the dual graph G^*

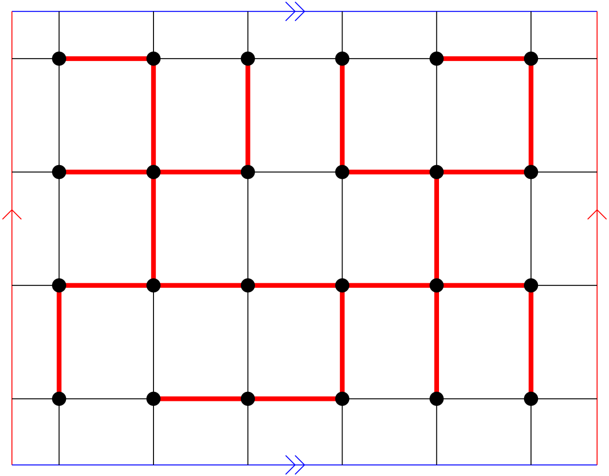


Tree-cotree partition - General

G embedded graph.
T a spanning tree of G
$C \subset E(G)$ cotree: C^* spanning tree of G^* disjoint from $E(T)^*$
X edges not in T or C. $X = \{e \in E(G) \mid e \notin E(T) \cup E(C)\}$

- (T, C, X) is a tree-cotree partition
- X has $2g$ edges (orientable) or g edges (non-orientable)
- (C^*, T^*, X^*) a tree-cotree partition of G^*
- for any $e \in X$, the cycle in $T + e$ is non-separating

Tree-cotree partition - Example



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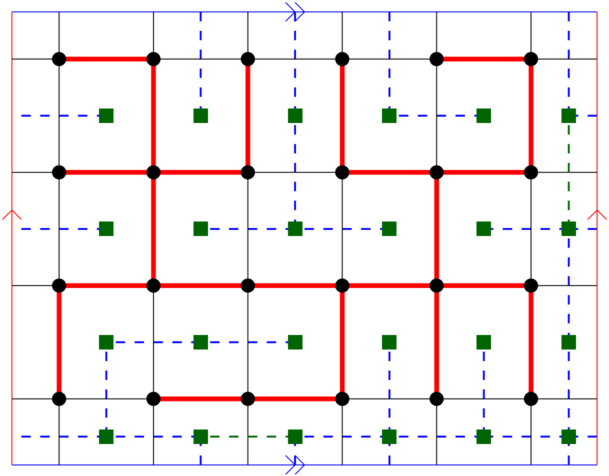

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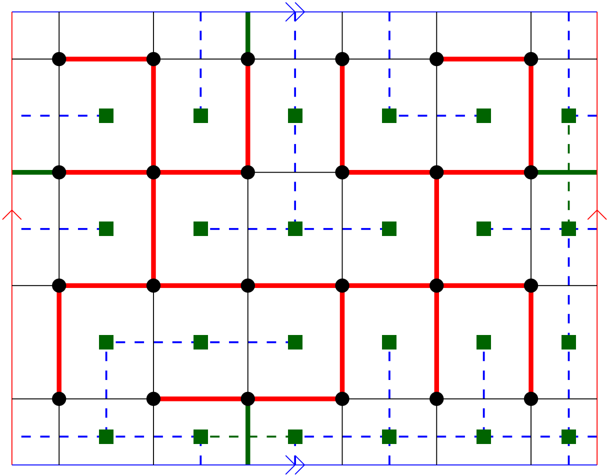


Tree-cotree partition - Cut graph

G embedded graph

$H \subset G$ a cut graph if G✄H is planar

- (T, C, X) is a tree-cotree partition of G
- $T \cup X$ is a cut graph: join faces according to C^*
- By duality, $C^* \cup X^*$ is a cut graph

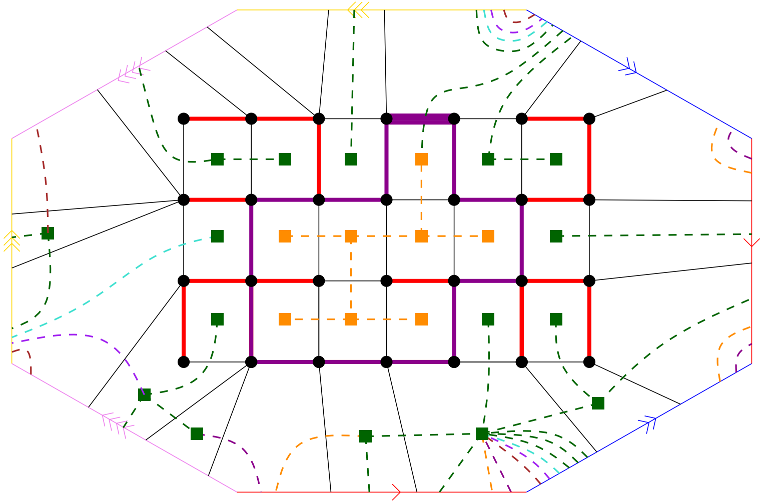
Tree-cotree partition - Nice loops

G embedded graph
(T, C, X) is a tree-cotree partition of G
$A = C \cup X$
$e \in A$

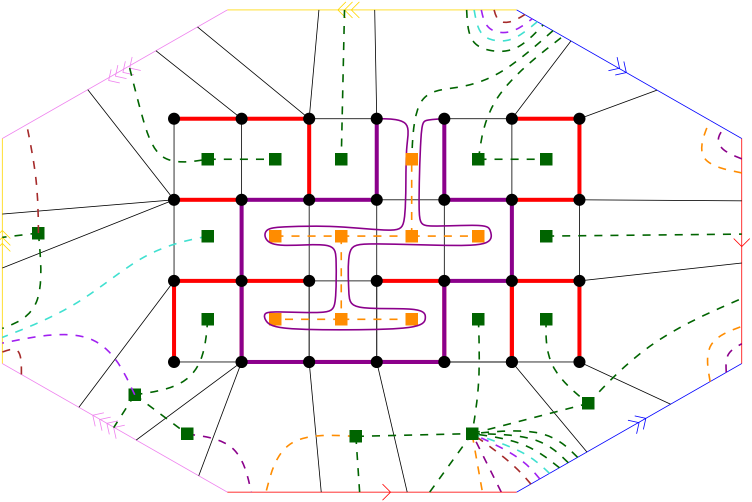
$\Rightarrow$ loop(T, e) contractible ifff $A^* - e^*$ has a tree component

- if loop(T, e) contractible $\Rightarrow$ loop(T, e) bounds a disk $D \Rightarrow$ $A - e$ contains a cotree of $G \cap D$
- if $A - e$ contains a cotree of $G \cap D \Rightarrow$ deform e along $A^* - e^*$ $\Rightarrow$ cycle homotopic to $A - e$ loop(T, e) disjoint from $A^* \Rightarrow$ loop(T, e) contractible

Nice loops - Contractible



Nice loops - Contractible

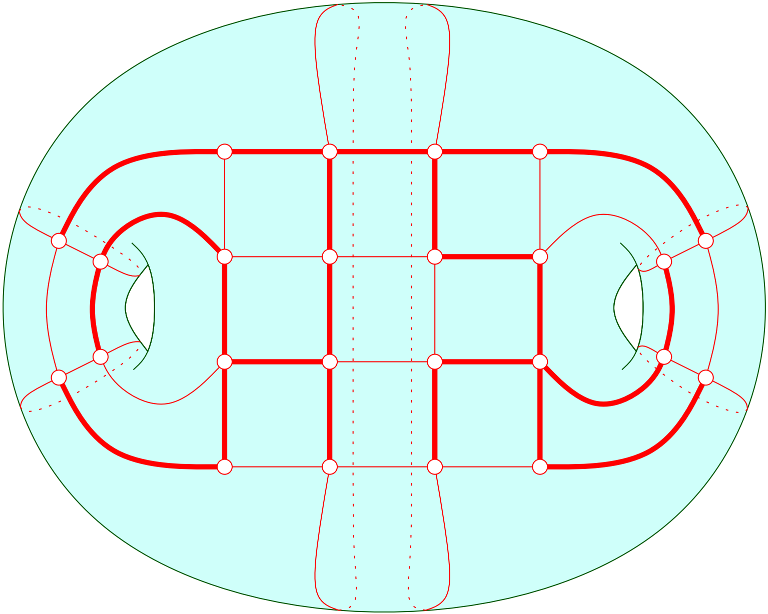


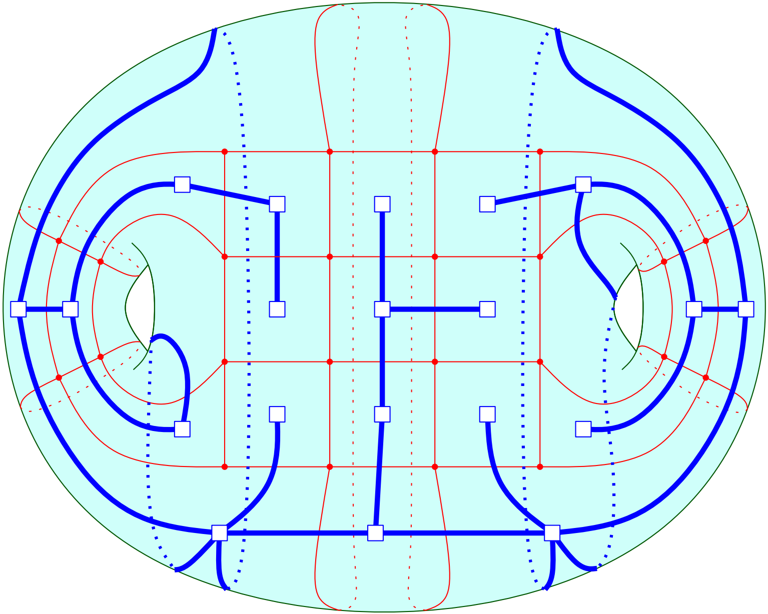
Tree-cotree partition - Nice loops

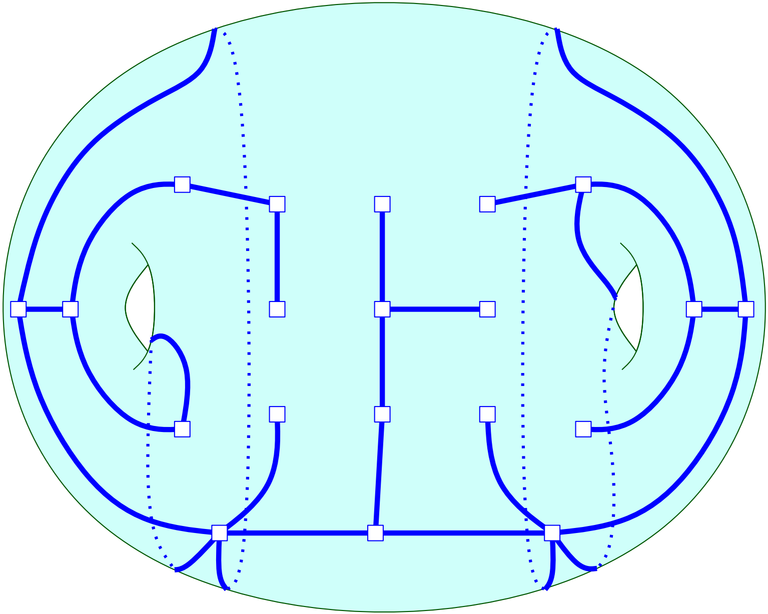
G embedded graph
(T, C, X) is a tree-cotree partition of G
$A = C \cup X$
$e \in A$

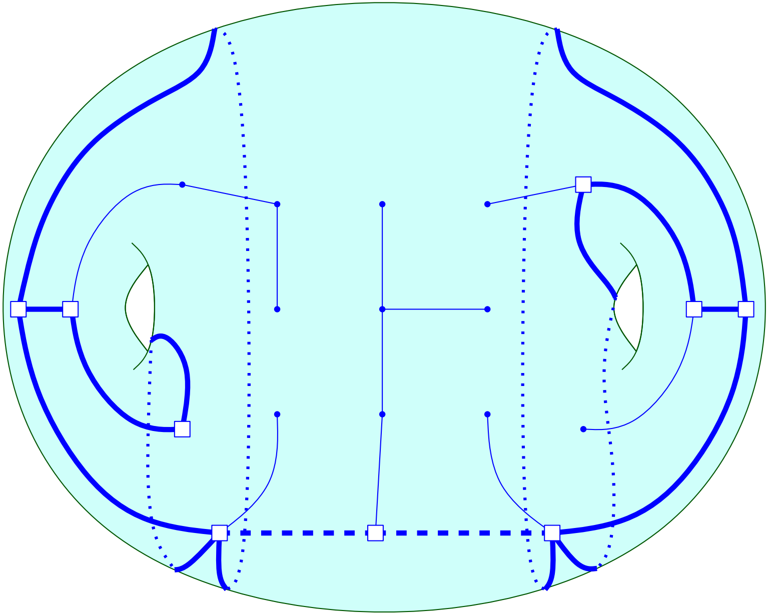
$\Rightarrow$ loop(T, e) separating ifff $A^* - e^*$ disconnected

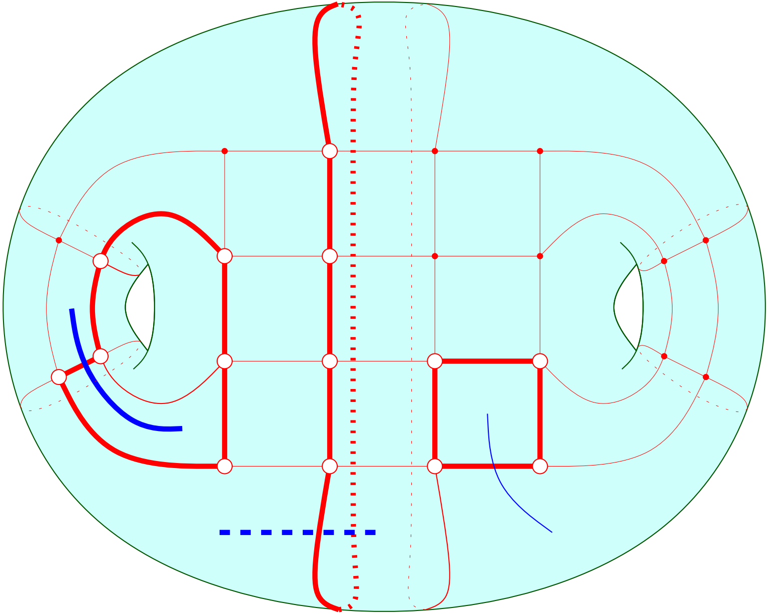
- $A^* - e^*$ gives a way to merge faces











Shortest non-contractible loop

G embedded graph
$x \in V(G)$
L_x contractible loops from x Compute shortest loop outside L_x

- compute shortest path tree T from x
- compute dual $A^* = G^* - E(T)^*$
- compute $B = \{e \in A \mid A^* - e^* \text{ has no tree-component}\}$
- compute

$$e = \arg\min_{uv \in B} \{d_T(x,u) + d_T(x,v) + |uv|\}$$

- return loop(T, e)

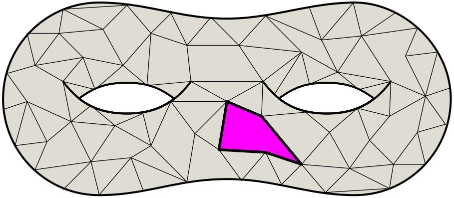
$\Rightarrow$ linear time per vertex x

Representation of some distances

Theorem

Let f be a specified face in an embedded graph G.
Preprocess G in $O(g^2 n \log n)$ time such that:

$$\textit{query } (u,v) \in V \times f \quad \xrightarrow{O(\log n) \textit{ time}} \quad \textit{distance } d_G(u,v)$$



- compute sp-tree (shortest path) at one vertex
- iteratively move to the neighbor in the face and update the sp-tree

Representation of some distances - Planar

Approach for planar graphs

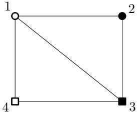
- compute sp-tree at one vertex of the face
- iteratively move to the neighbor in the face and update the sp-tree
- efficient dynamic data structures to detect what edges come in and out
- reminiscence of *kinetic* data structures
- use of tree-cotree decomposition
- each (directed) edge appears in a contiguous family of sp-trees (via crossing argument)
- persistence

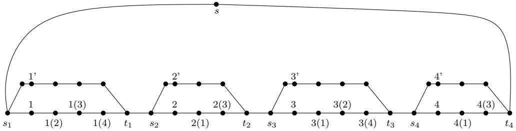
Parity of crossings of cycles for separating cycles

- α and β cycles in G
- $cr(\alpha, \beta) = \min cr(\alpha', \beta)$ over all tiny deformations α' of α
- $cr_2(\alpha, \beta) = cr(\alpha, \beta) \mod 2$
- computing $cr_2(\alpha, \beta)$ is easy
  - invariant under tiny deformations
- useful to work over $\mathbb{Z}_2$-homology
- α separating iff $cr_2(\alpha, \cdot) = 0$
- $cr_2 : H_1 \times H_1$ is well-defined and bilinear

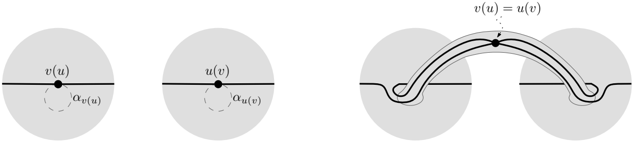
Shortest separating cycle

- max independent set reduces to:
  shortest cycle in planar graph with forbidden pairs





- surgery to represent the forbidden pairs



- separating cycle $\Leftrightarrow$ crosses any closed curve even nb of times

Conclusions

- A taste of the algorithmic problems for embedded graphs
- A taste of the techniques

- Gap theory-practice
- Representation-free algorithms
- H-minor-free graphs
- Simple simplicial complexes, like $\beta_i = O(1)$ for all i.