Scribability Problems for Polytopes

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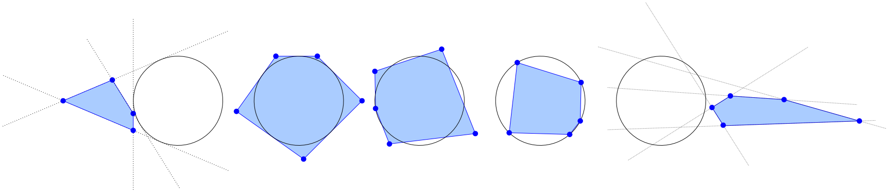
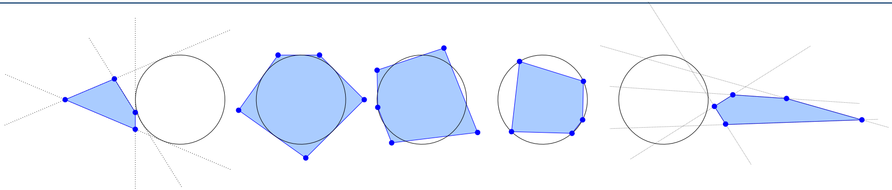
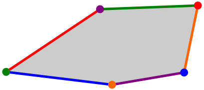
joint work with
Hao Chen (FU Berlin ⟶ TU Eindhoven)

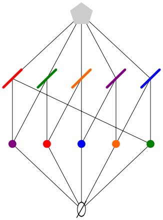


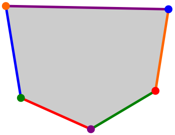


Scribability Problems

Study realizability of polytopes when
the position of their faces relative to the sphere is constrained.







Classical scribability problems

Jakob Steiner (according to Wikipedia)

Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander (1832)

77) Wenn irgend ein convexes Polyeder gegeben ist läßt sich dann immer (oder in welchen Fällen nur) irgend ein anderes, welches mit ihm in Hinsicht der Art und der Zusammensetzung der Grenzflächen übereinstimmt (oder von gleicher Gattung ist), in oder um eine Kugelfläche, oder in oder um irgend eine andere Fläche zweiten Grades beschreiben (d.h. daß seine Ecken alle in dieser Fläche liegen oder seine Grenzflächen alle diese Fläche berühren)?

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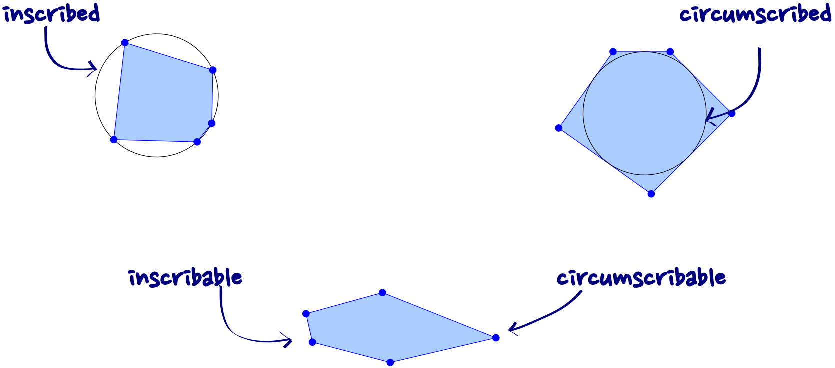
- Is every (3-dimensional) polytope inscribable?
- If not, in which cases?
- What about circumscribable?
- What about other quadrics?

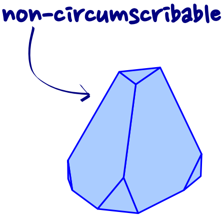
Inscribable & circumscribable polytopes

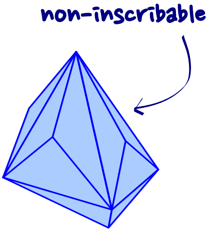


Circumscribability

Steinitz 1928
Über isoperimetrische Probleme bei konvexen Polyedern

- P is circumscribable $\Leftrightarrow$ P^* is inscribable
- There exist infinitely many non-circumscribable 3-polytopes.





Igor Rivin

Rivin 1996
A characterization of ideal polyhedra in hyperbolic 3-space

A 3-polytope P is circumscribable if and only if there exist numbers $\omega(e)$ associated to the edges e of P such that:

- $0 < \omega(e) < \pi$,
- $\sum_{e \in F} \omega(e) = 2\pi$ for each facet F of P, and
- $\sum_{e \in C} \omega(e) > 2\pi$ for each simple circuit C which does not bound a facet.

Inscribability on other quadrics


Jeffrey Danciger


Sara Maloni


Jean-Marc Schlenker

Danciger, Maloni & Schlenker 2014
Polyhedra inscribed in a quadric

A 3-polytope is inscribable on a hyperboloid or a cylinder if and only if it is inscribable and Hamiltonian.

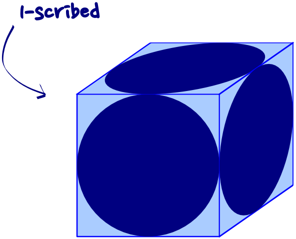
More classical scribability problems

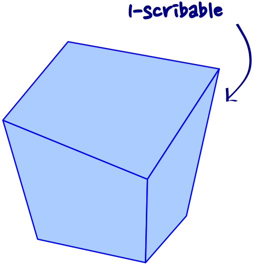
B. Grünbaum

Grünbaum & Shephard 1987
Some problems on polyhedra

For which k and d is every d-polytope k-scribable?

G. Shephard





Schulte 1987
Analogues of Steinitz's theorem about non-inscribable polytopes

Except for $d \leq 2$ or $d = 3$ and $k = 1$, there are d-polytopes that are not k-scribable.

Egon Schulte

The only case that has resisted all efforts so far is the escribability in three dimensions. Somehow it is strange that all higher-dimensional analogues turn out to be solvable, while the 'elementary' three-dimensional case of escribability seems to be intractable.

Edgescribability


P. Koebe

E. Andreev


W. Thurston

Koebe 1936 – Andreev 1970 – Thurston 1982
The geometry and topology of 3-manifolds

Every 3-polytope is 1-scribable.

Koebe – Andreev – Thurston
The circle packing Theorem

Every planar graph is representable by a circle packing (with a dual circle packing).

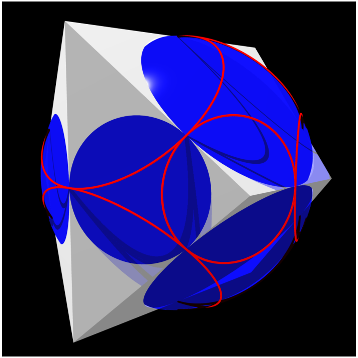


Figure from Wikipedia by D. Eppstein

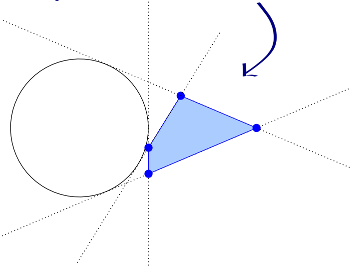
B. Grünbaum

Grünbaum & Shephard 1987
Some problems on polyhedra

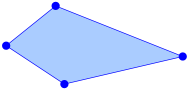
For which k and d is every d-polytope weakly k-scribable?

G. Shephard

weakly circumscribed



weakly circumscribable



Schulte 1987
Analogues of Steinitz's theorem about non-inscribable polytopes

Except for $d \leq 2$ or $d = 3$ and $k = 1$, if $0 \leq k \leq d-3$ there are d-polytopes that are not weakly k-scribable.

Egon Schulte

The attempt to generalize our methods immediateiy reveals the main difference between (m,d)-scribability and weak (m,d)-scribability. In fact, a polytope with all m-faces tangent to $\mathbb{S}^{d-1}$ has necessarily an interior point in the open unit-ball, while this need not be true if only the affine hulls of all m-faces are tangent to $\mathbb{S}^{d-1}$. Therefore, Theorem 1 might fail.

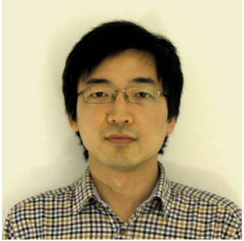
New scribability problems

(i, j)-scribability

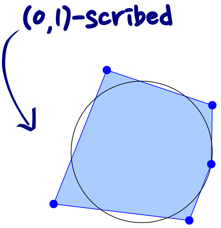
Chen & P. 2015+
Scribability problems for polytopes

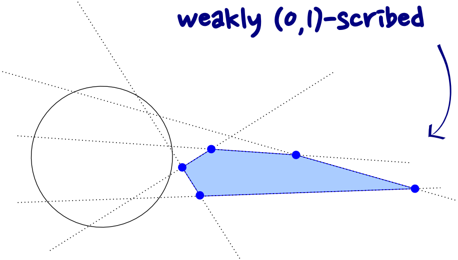
For which i, j and d is every d-polytope (i, j)-scribable?

P is (weakly) (i, j)-scribed if
all (affine hulls of) i-faces *avoid* the sphere and
all (affine hulls of) j-faces *cut* the sphere



Hao Chen





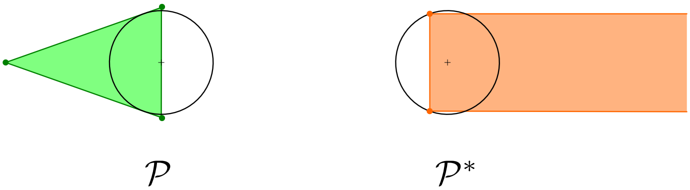
Some properties of (i,j)-scribability

- strongly (i,j)-scribed $\Rightarrow$ weakly (i,j)-scribed

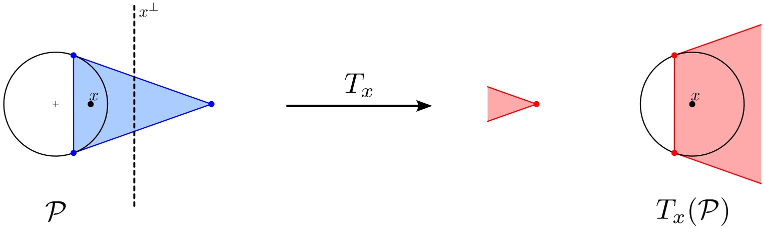
In the strong and weak sense:

- (k,k)-scribed $\Leftrightarrow$ k-scribed (classic definition)
- (i,j)-scribed $\Rightarrow$ (i',j')-scribed for $i' \leq i$ and $j' \geq j$
- (i,j)-scribed $\Rightarrow$ polar $(d-1-j, d-1-i)$-scribed
- (i,j)-scribed $\Rightarrow$ facets (i,j)-scribed
- (i,j)-scribed $\Rightarrow$ vertex figures $(i-1, j-1)$-scribed

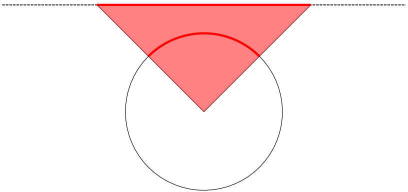
Polarity is not nice for polytopes. . .

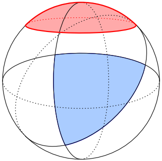


. . . and cannot always be saved with a projective transformation . . .



. . . but it is for cones or spherical polytopes!



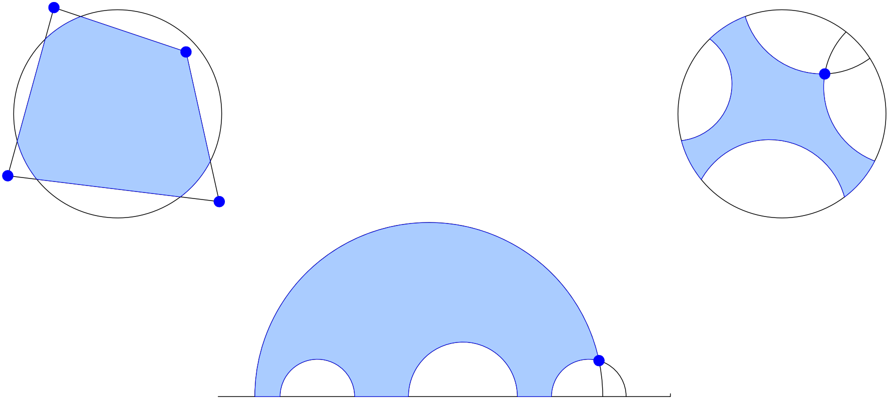


Definition

$P \subset \mathbb{S}^d$ spherical polytope, F face of P. Then

- F *strongly cuts* $\mathbf{B}^d$ if $\operatorname{relint}(F) \cap \mathbf{B}^d \neq \emptyset$;
- F *weakly cuts* $\mathbf{B}^d$ if $\operatorname{span}(F) \cap \mathbf{B}^d \neq \emptyset$;
- F *weakly avoids* $\mathbf{B}^d$ if $\operatorname{span}(F) \cap \operatorname{int} \mathbf{B}^d = \emptyset$;
- F *strongly avoids* $\mathbf{B}^d$ if there is a supporting hyperplane H of P such that $F = H \cap P$ and $\mathbf{B}^d \subset H^-$.

Hyperbolic polyhedra

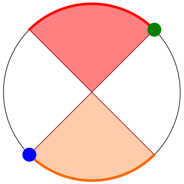


Facets intersecting the sphere are hyperbolic polyhedra

A warm-up: weak (i, j)-scribability

A weakly inscribable polytope

A Euclidean weakly inscribable polytope is also strongly inscribable. But this is no longer true with spherical polytopes!



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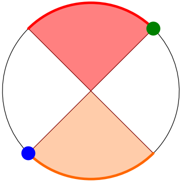
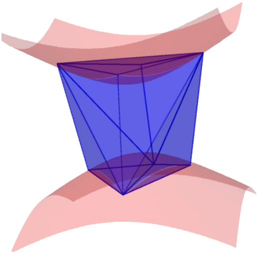
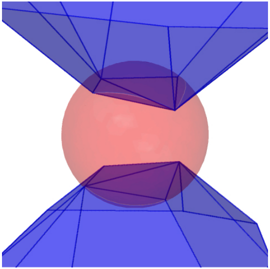
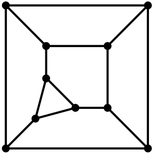


For example, the triakis tetrahedron is weakly inscribable!

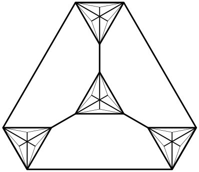




This polytope is not weakly inscribable

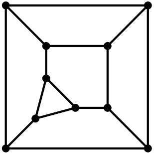


This polytope is not weakly circumscribable



Two non-weakly inscribable polytopes

This polytope is not weakly inscribable



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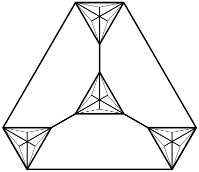
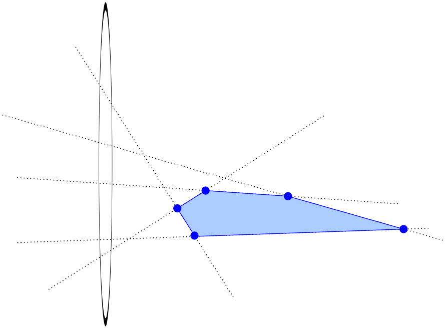


Theorem (Chen & P. 2015+)

Except for $d \leq 2$ or $d = 3$ and $k = 1$,
there are d-polytopes that are not weakly (k, k)-scribable.

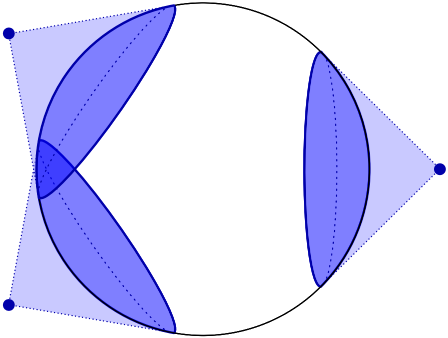
Theorem (Chen & P. 2015)

Every d-polytope is weakly (i,j)-scribable for $0 \leq i < j \leq d-1$.



Strong (i,j)-scribability: cyclic polytopes

From polytopes to spheres



Points surrounding the sphere are arrangements of spheres

$B_1, \ldots, B_n$ form a k-ply system

$\Updownarrow$

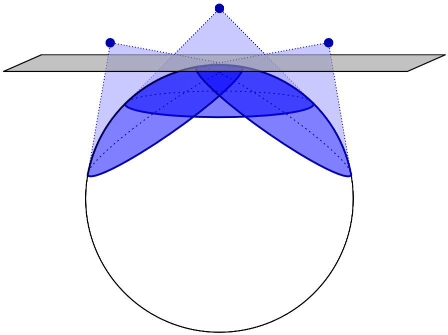
no point belongs to the interior of more than k balls.

$\Updownarrow$

No more than k “satellites” can be linearly separated from $\mathbb{S}^d$

$\Updownarrow$

The convex hull of every $(k+1)$-set intersects $\mathbb{S}^d$



The Sphere Separator Theorem

Gary Miller

Shang-Hua Teng

William Thurston

Stephen Vavasis

Miller, Teng, Thurston & Vavasis 1997
Separators for Sphere-Packings and Nearest Neighbor Graphs

The intersection graph of a k-ply system of n caps in $\mathbb{S}^d$ can be separated into two disjoint components of size at most $\frac{d+1}{d+2}n$ by removing $O(k^{1/d}n^{1-1/d})$ vertices.

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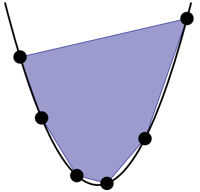
- Wlog 0 is the centerpoint of centers of the B_i
- Take a random linear hyperplane
- Probability it hits a cap depends on its surface
- Since k-ply it can be bounded



Moment curve

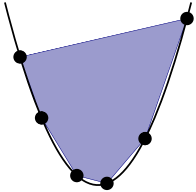
$$\gamma : t \mapsto (t, t^2, \dots, t^d)$$

Cyclic polytope

$$\mathcal{C}_d(n) = \operatorname{conv}\left\{\gamma(t_1), \dots, \gamma(t_n), t_1 < t_2 < \cdots < t_n\right\}$$

Combinatorial type:

- Independent of the t_i's,
- Same for γ any order d curve.
  For example: Trig. moment curve: $\gamma : t \mapsto (\sin t, \cos t, \dots, \sin kt, \cos kt)$
- Simplicial
- $\left\lfloor \frac{d}{2} \right\rfloor$-neighborly: every subset of at most $\left\lfloor \frac{d}{2} \right\rfloor$ vertices forms a face.



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- Simplicial
- $\left\lfloor \frac{d}{2} \right\rfloor$-neighborly: every subset of at most $\left\lfloor \frac{d}{2} \right\rfloor$ vertices forms a face.

Theorem (Upper bound theorem [McMullen 1970])

If P is a d-polytope with n vertices

$$f_i(P) \leq f_i(\mathcal{C}_d(n))$$

with equality if and only if P is simplicial and neighborly.

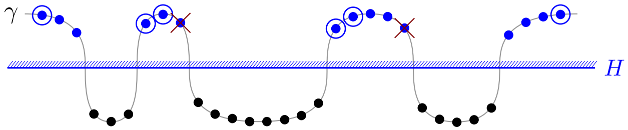
Lemma (Sturmfels 1987)

For even d, every cyclic polytope comes from a order d curve

(The keyword here is oriented matroid rigidity)

Lemma

If d even, every k-set of $\mathcal{C}_d(n)$ with $k \geq 3\frac{d}{2} - 1$ contains a facet.



Theorem

If $d \geq 4$ even and $n \gg 0$ is large enough, $\mathcal{C}_d(n)$ is not $(1, d-1)$-scribable.

Proof.

Assume it was $(1, d-1)$-scribed:

- Every $k(= 3\frac{d}{2} - 1)$-set contains a facet
- $\Rightarrow$ intersects $\mathbb{S}^d$
- $\Rightarrow$ induces a k-ply system
- $\Rightarrow$ intersection graph has small separators

- Intersection graph are edges avoiding the sphere
- $\Rightarrow$ complete graph!

$\square$

Our proofs only extend partially to neighborly polytopes.

Lemma

For $d \geq 4$ and $n \gg 0$, a k-neighborly polytope with n vertices is not $(1, k)$-scribable.

This suffices to show that:

Corollary

For $d \geq 4$ and $1 \leq k \leq d-2$, there is a non k-scribable f-vector.

Question

Is there a non-inscribable f-vector? For example, that of a dual-to-neighborly polytope with many vertices?

What about stacked polytopes?

Most stacked polytopes are not inscribable [Gonska & Ziegler 2013]

Theorem

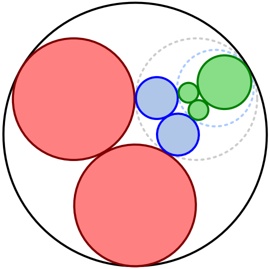
For every $0 \leq k \leq d-3$, there is a stacked d-polytope that is not k-scribable.

But...

Theorem

Every stacked d-polytope is circumscribable and ridgescribable.

Hence, truncated polytopes are always edgescribable!



What now?

Question

Is every d-polytope $(0, d-1)$-scribable?

We don't think so but . . .

- true in dimension ≤ 3
- cyclic polytopes are $(0, d-1)$-scribable
- stacked polytopes are $(0, d-1)$-scribable

Open problem

Question

Is every d-polytope $(0, d-1)$-scribable?

We don't think so but . . .

- true in dimension ≤ 3
- cyclic polytopes are $(0, d-1)$-scribable
- stacked polytopes are $(0, d-1)$-scribable

Thank you!