Quasiconformal distortion of projective maps and discrete conformal maps

with Stefan Born and Ulrike Bücking

arXiv:1505.01341

Bobenko, Pinkall, S
Discrete conformal maps and ideal hyperbolic polyhedra
Geom. Topol. 19-4 (2015), 2155-2215

S, Schröder, Pinkall
Conformal equivalence of triangle meshes
ACM Transactions on Graphics 27:3 (2008)

1 Discrete conformal maps

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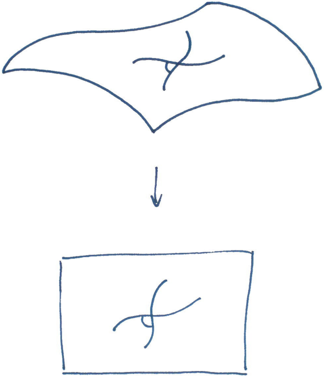
- *conformal* means *angle preserving*
- lengths scaled by *conformal factor* independent of direction

$$\|df_p(v)\| = e^{u(p)}\,\|v\|$$

- looks like a similarity transformation when zooming in



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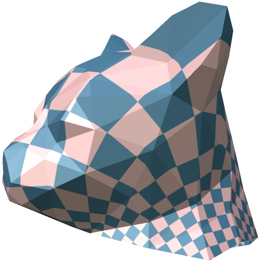
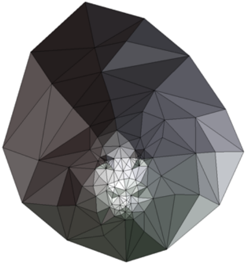




1 Discrete conformal maps: scale factors

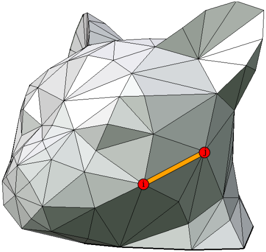
Definition (Luo 2004)

Two triangulated surfaces are discretely conformally equivalent, if

(i) triangulations are combinatorially equivalent

(ii) edge lengths ℓ_{ij} and $\tilde{\ell}_{ij}$ related by

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i+u_j)}\ell_{ij}$$

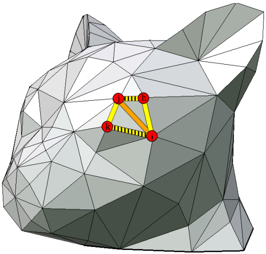


- Leads to rich theory with connections to hyperbolic geometry.

1 Discrete conformal maps: length cross ratio

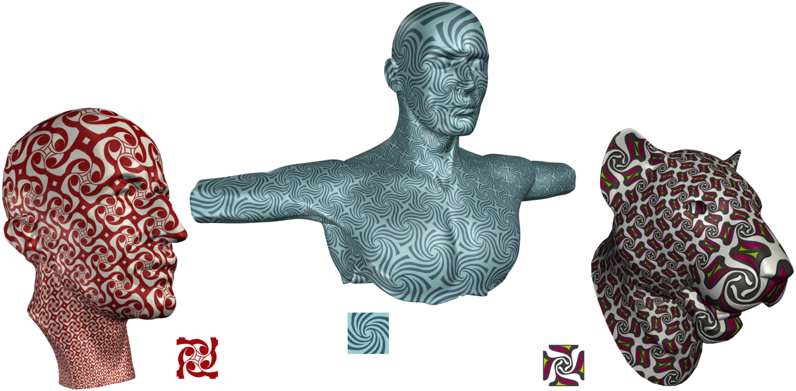
For interior edges *ij* define
length cross ratio

$$\mathrm{lcr}_{ij} = \frac{\ell_{ih}\,\ell_{jk}}{\ell_{hj}\,\ell_{ki}}$$



Theorem
ℓ, $\tilde{\ell}$ *discretely conformally equivalent*
$\Longleftrightarrow \widetilde{\mathrm{lcr}} = \mathrm{lcr}$

1 Discrete conformal maps: Examples



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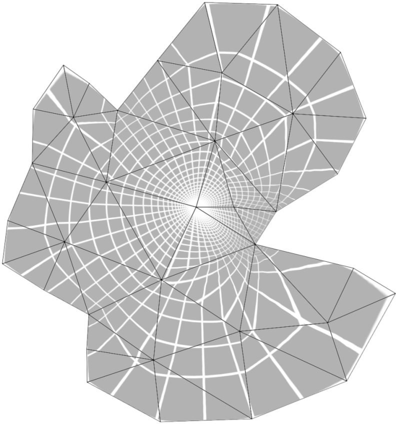

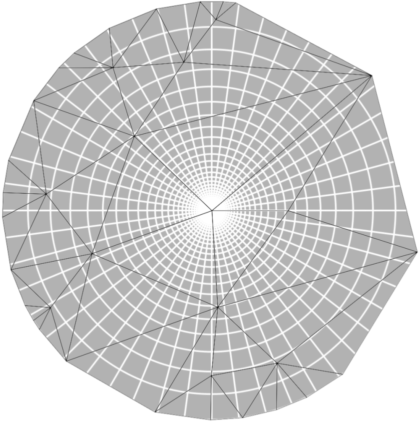

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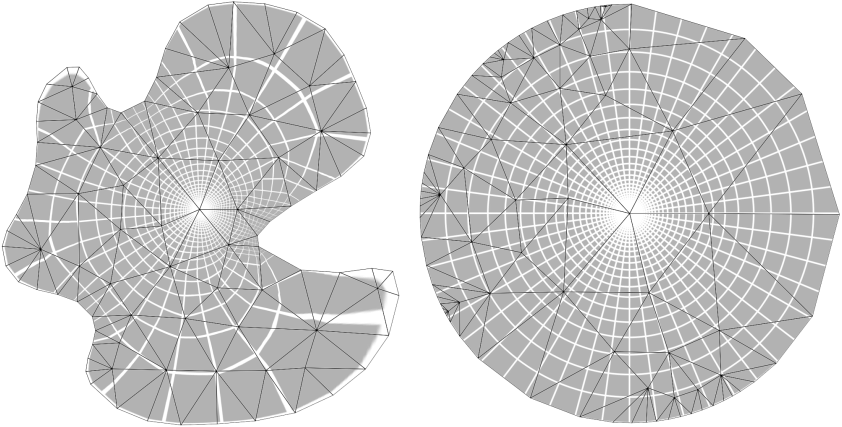

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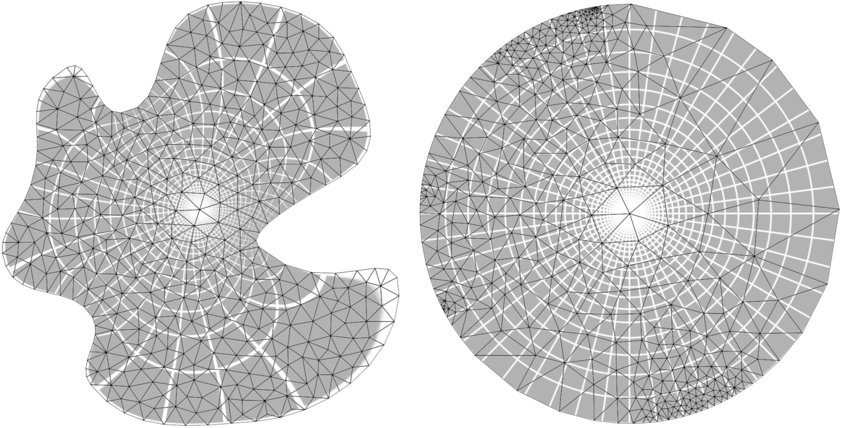

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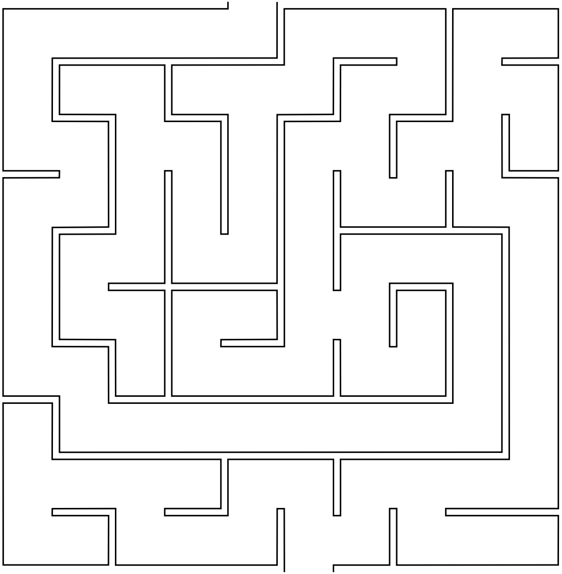

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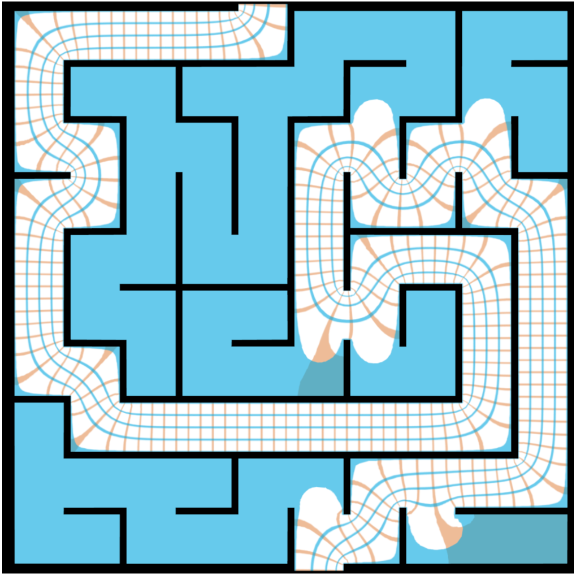

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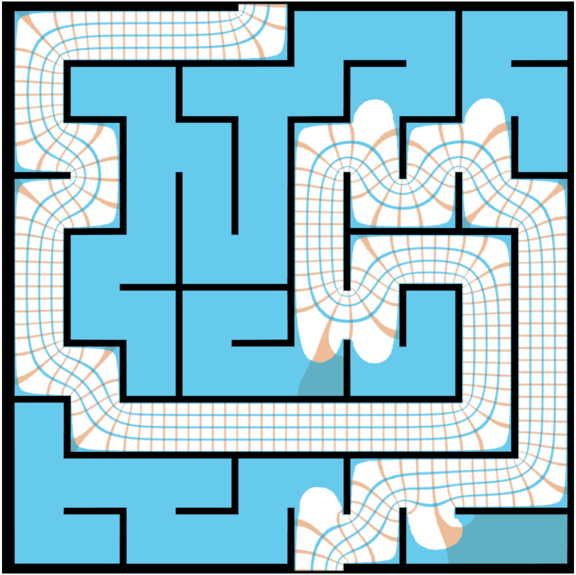

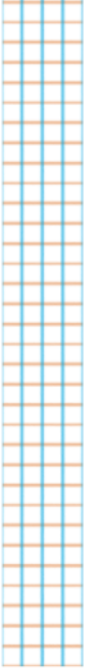

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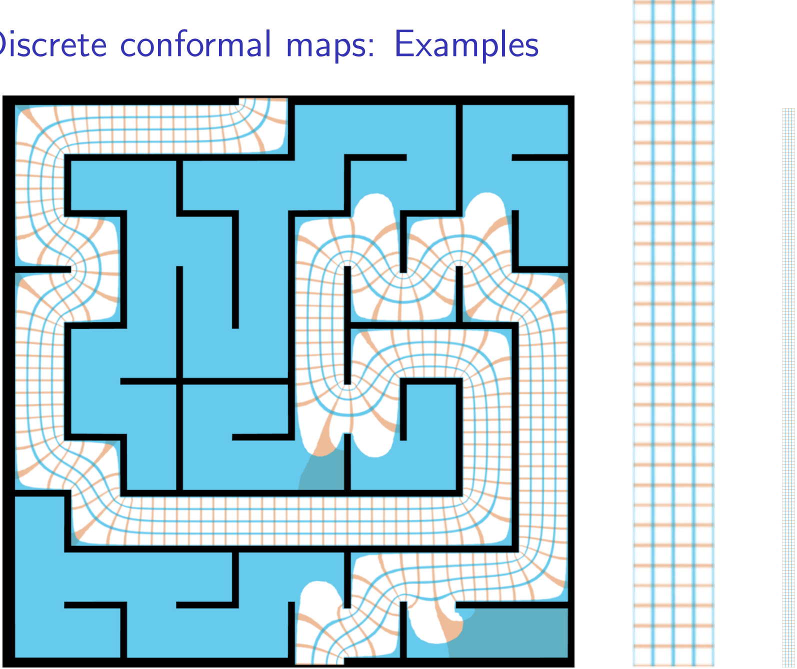

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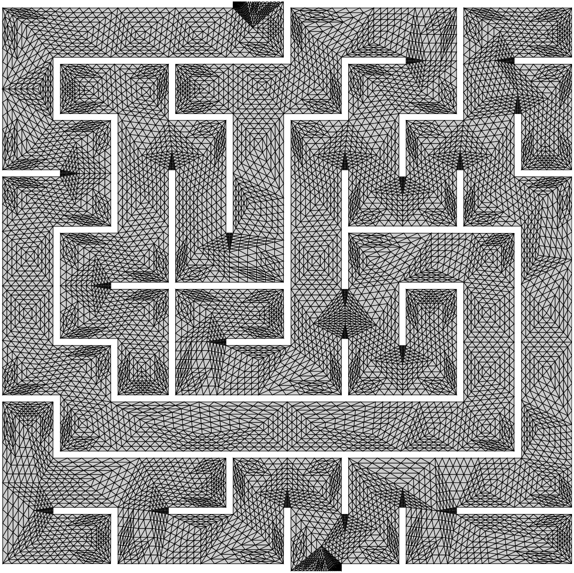
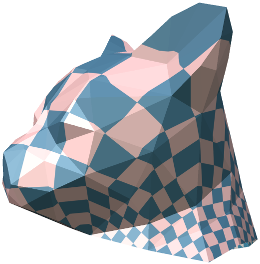
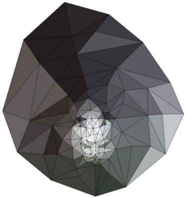


1 Discrete conformal maps: Interpolation

- How to interpolate over triangles?
- piecewise linear always works
- better: *circumcircle preserving piecewise projective (cpp) maps*



piecewise linear



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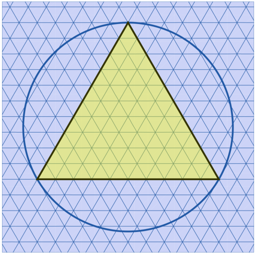

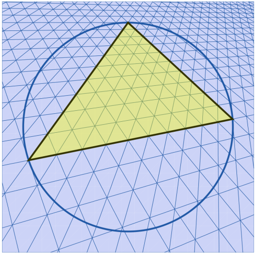

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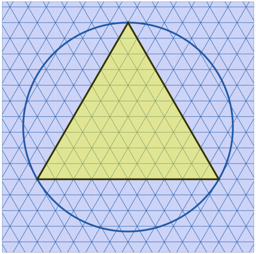

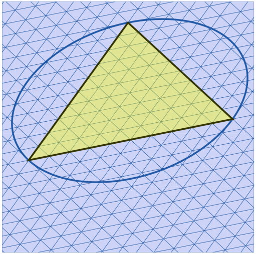

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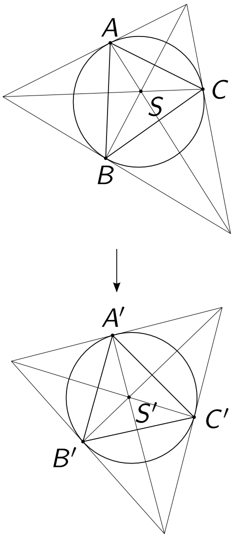

S, S': symmedian (Lemoine, Grebe) points

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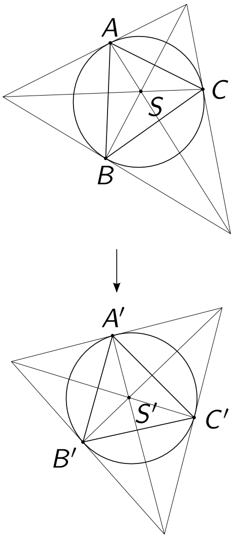
Theorem
cpp maps fit together continuously across edges
$\Longleftrightarrow$
triangulations are discretely conformally equivalent



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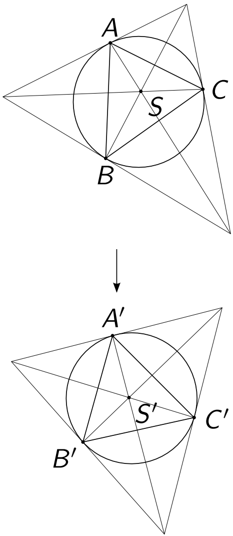
discrete conformal map:
simplicial map, cpp on triangles



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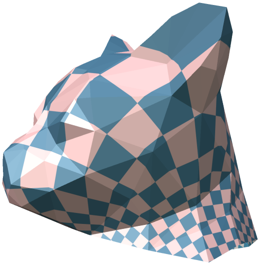

cpp

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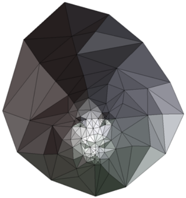
- cpp interpolation is "visibly smoother"



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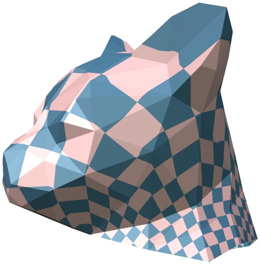

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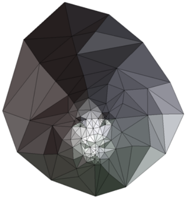

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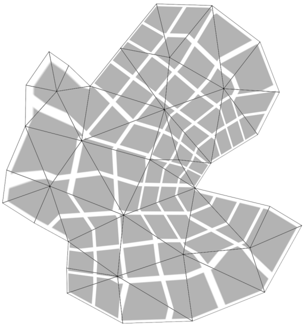

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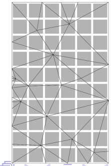

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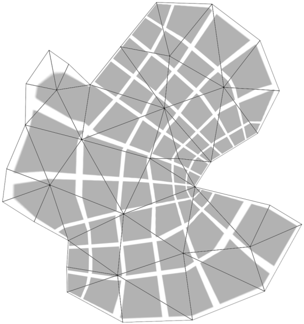

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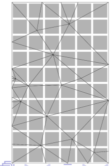

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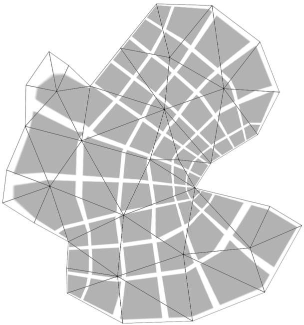

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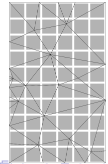
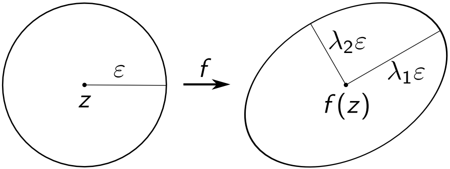


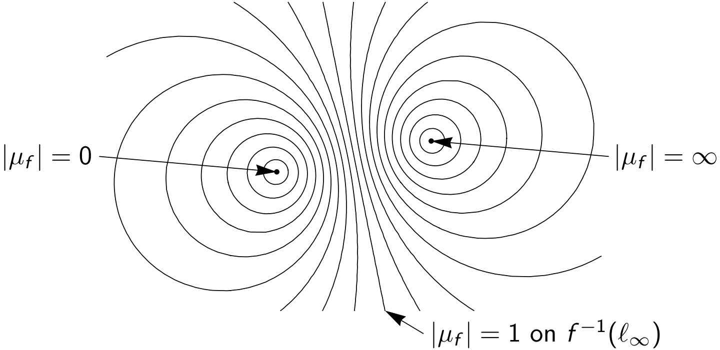
- cpp interpolation is “visibly smoother”
- **Why?**
- Lower quasiconformal distortion?

2 Quasiconformal distortion



- $0 \le \lambda_2 \le \lambda_1$ singular values of df_z
- $D_f(z) = \pm\dfrac{\lambda_1}{\lambda_2}$, sign depends on orientation
- $|\mu_f| = \dfrac{D_f - 1}{D_f + 1}$ modulus of Beltrami differential
- $|\mu_f| = \begin{cases} 0 & \text{where } df \text{ is conformal} \\ 1 & \text{where } df \text{ is singular} \\ \infty & \text{where } df \text{ is anticonformal} \end{cases}$

3 Distortion of a projective map

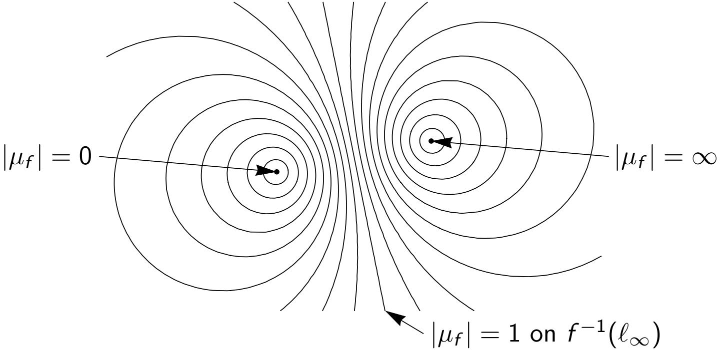


Theorem

(i) *If projective map $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ is not affine, contourlines of $|\mu_f|$ form a hyperbolic pencil of circles.*

(ii) *This hyperbolic pencil of circles is mapped to another hyperbolic pencil of circles.*

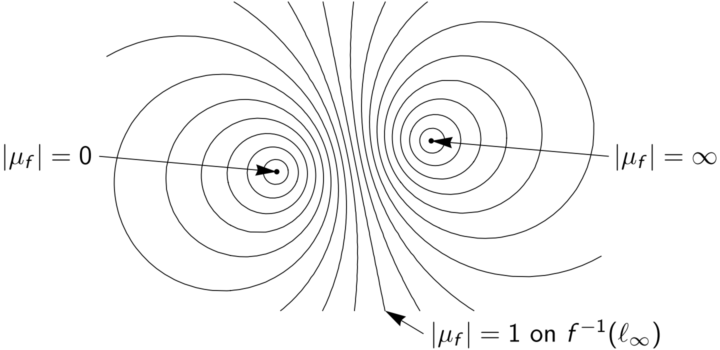
3 Distortion of a projective map



Corollary

If f is orientation preserving on triangle ABC, then $\max_{z \in ABC} |\mu_f(z)|$ is attained at A, B, or C.

3 Distortion of a projective map & circles mapped to circles



- Which circles are mapped to circles by a projective map f?

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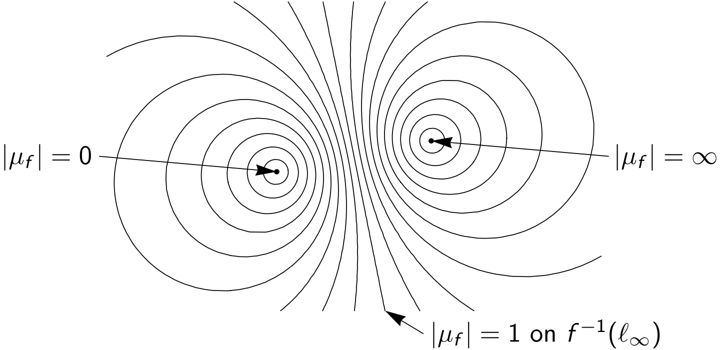
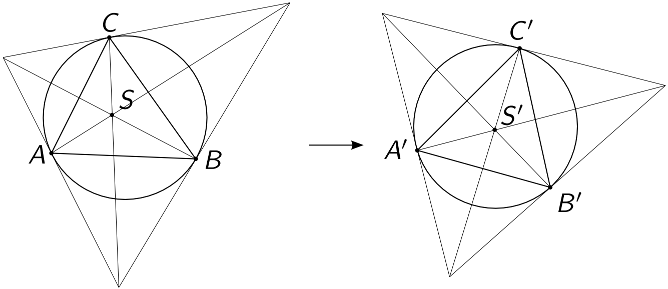


- Which circles are mapped to circles by a projective map f?

Theorem

- *If* $f \in \mathrm{Sim}$*: all circles*
- *If* $f \in \mathrm{Aff} \setminus \mathrm{Sim}$*: no circle*
- *If* $f \notin \mathrm{Aff}$*: exactly one hyperbolic pencil of circles*

4 Distortion of circumcircle preserving projective map



Theorem

If $f : ABC \to A'B'C'$ is a cpp map, then

$$|\mu_f(A)| = |\mu_f(B)| = |\mu_f(C)| = |\mu_h|,$$

where h is the affine map $ABC \to A'B'C'$.

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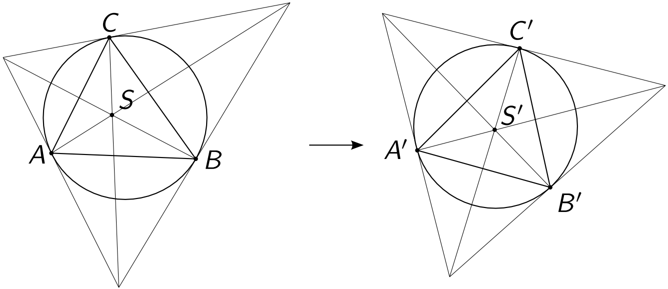

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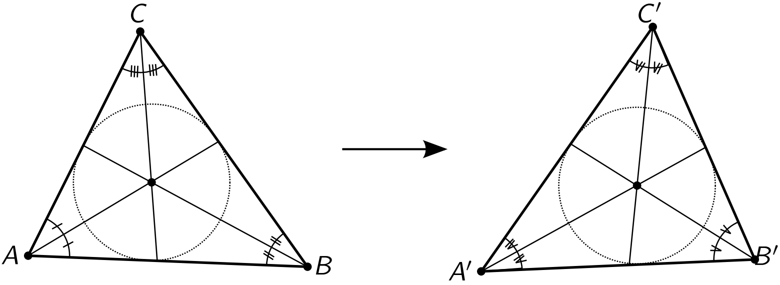
- cpp interpolation better than linear interpolation (except at vertices)

5 Angle bisector preserving projective map



Theorem

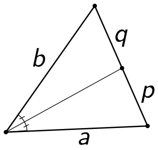
Of all projective maps $ABC \to A'B'C'$*, the* *angle bisector preserving projective map (app map)* *simultaneously minimizes* $|\mu_f(A)|$, $|\mu_f(B)|$, $|\mu_f(C)|$.

5 Angle bisector preserving projective map

Theorem

Two triangulations are discretelely conformally equivalent
$\Leftrightarrow$ *app maps are continuous across edges.*

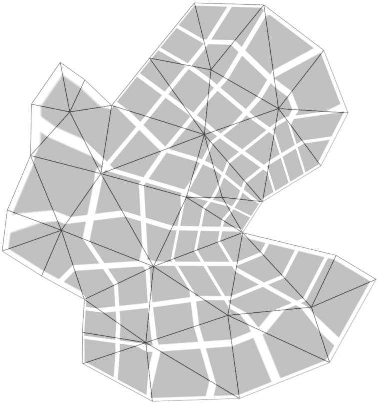
- follows from angle bisector theorem

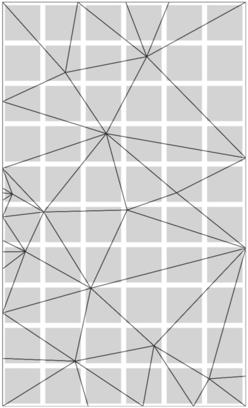


$$\frac{a}{b} = \frac{p}{q}$$

5 Angle bisector preserving projective map

Which interpolation looks best?





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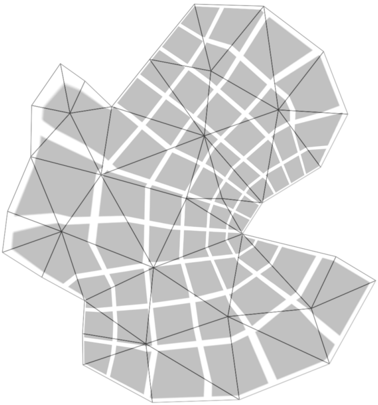

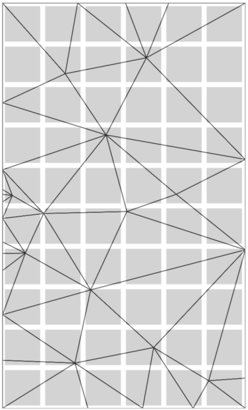

app

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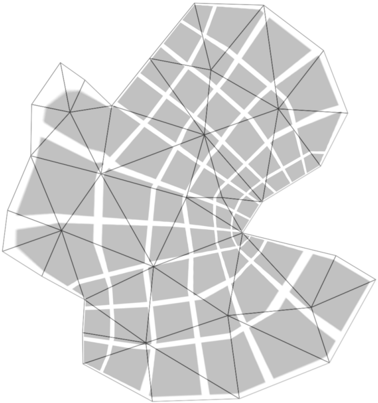

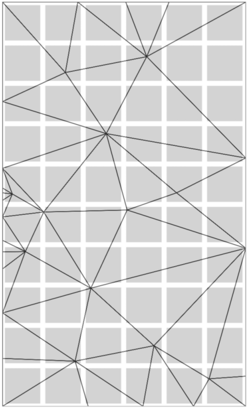

cpp

6 A 1-parameter familiy of projective interpolation schemes

- Barycenter has barycentric coordinates $[1, 1, 1]$
- Incircle center has barycentric coordinates $[a, b, c]$
- Symmedian point has barycentric coordinates $[a^2, b^2, c^2]$

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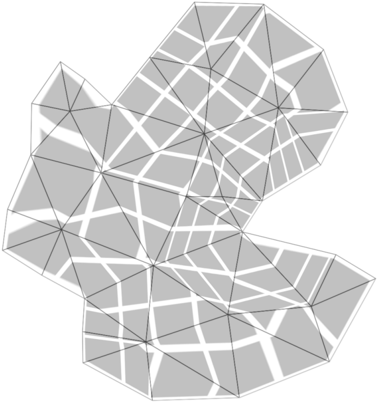
- Exponent-t-center has barycentric coordinates $[a^t, b^t, c^t]$

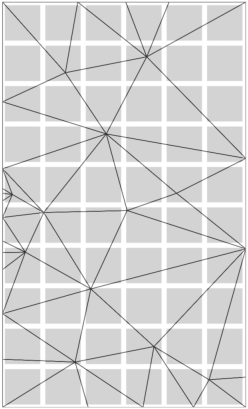
Theorem

The projective maps that map exponent-t-centers to exponent-t-centers fit together continuously across edges if, and for $t \neq 0$ only if, the triangulations are discretely conformally equivalent.

6 A 1-parameter familiy of projective interpolation schemes



$t = -1.0$



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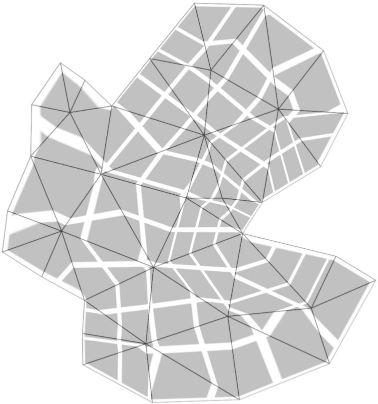
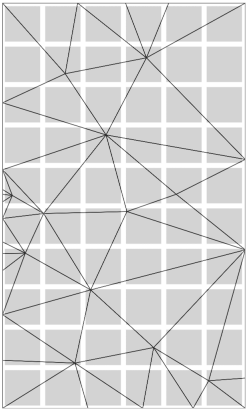
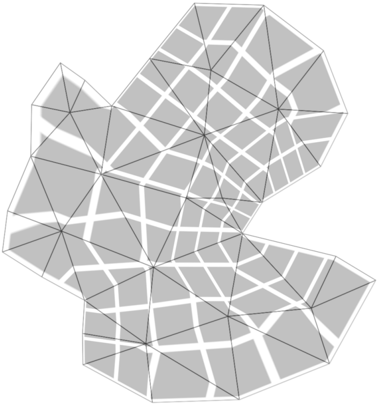


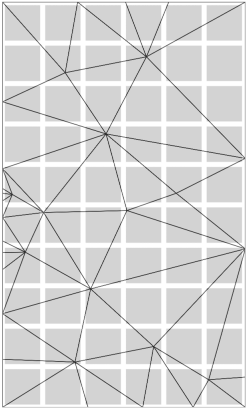


$t = -0.5$

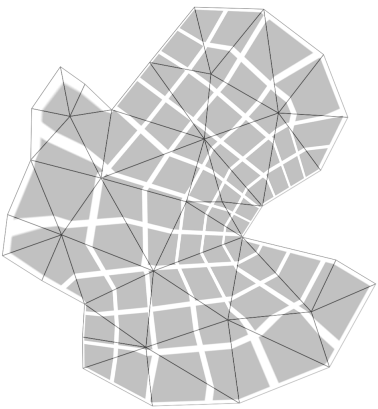
6 A 1-parameter familiy of projective interpolation schemes

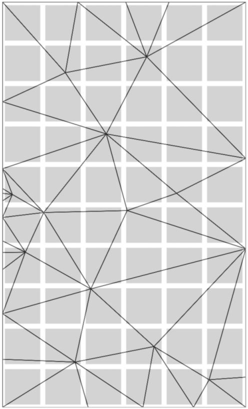


$t = 0.0$ (linear)



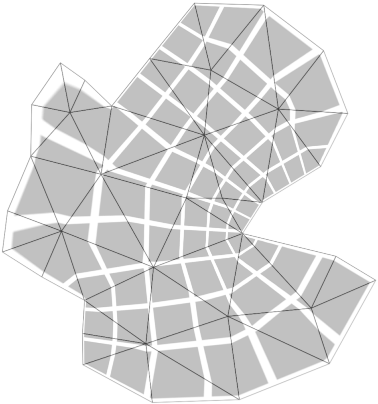
6 A 1-parameter familiy of projective interpolation schemes



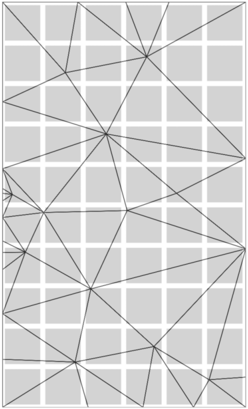


$t = 0.5$

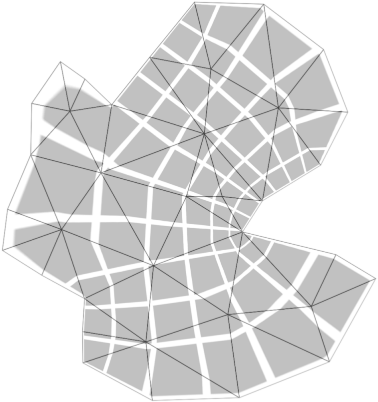
6 A 1-parameter familiy of projective interpolation schemes

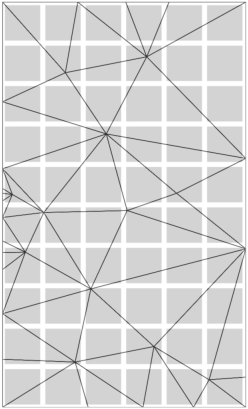


$t = 1.0$ (app)



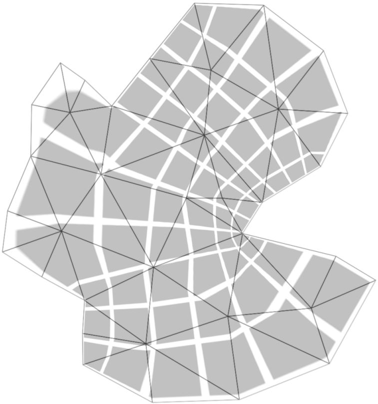
6 A 1-parameter familiy of projective interpolation schemes

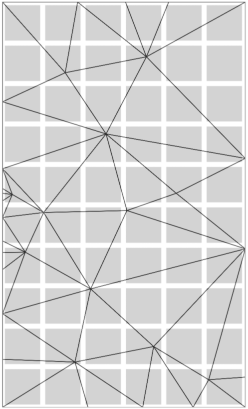




$t = 1.5$

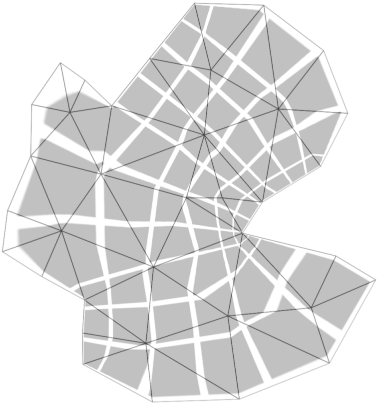
6 A 1-parameter familiy of projective interpolation schemes

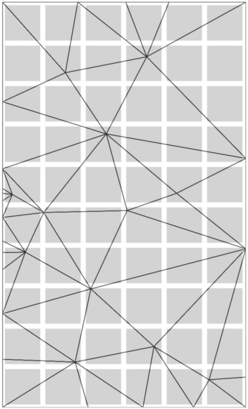




$t = 2.0$ (cpp)

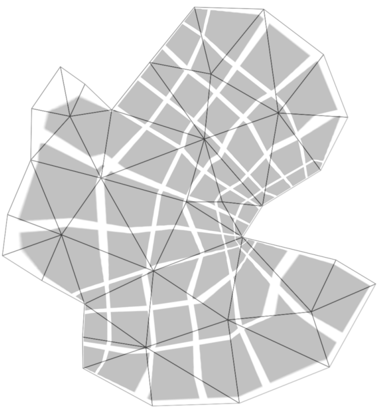
6 A 1-parameter familiy of projective interpolation schemes

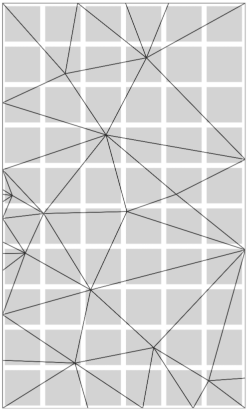




$t = 2.5$

6 A 1-parameter familiy of projective interpolation schemes





$t = 3.0$