Intrinsic simplices on spaces of nearly constant curvature

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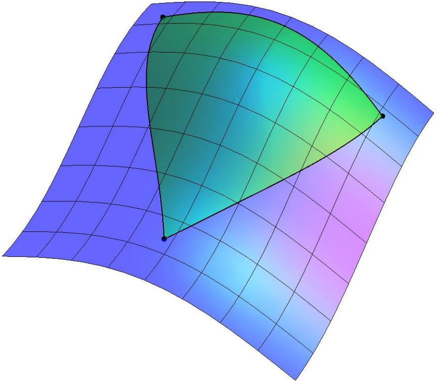
Workshop on computational geometry in non-Euclidean spaces

Intrinsic simplices on Riemannian manifolds

Motivation: Generic triangulation criteria

- intrinsic setting
- explicit quality requirements
- Arbitrary dimension



Modeling simplices: quality

Non-degeneracy by quality requirements depending on curvature manifold

- SOCG: Model simplices on Euclidean space
  - To tangent space via exponential map
  - Almost flat
- Here: Model simplices using space of constant curvature
  - Map to space of constant curvature
  - Curvature is (locally) nearly constant
  - Towards adaptive sampling

New quality measures for spaces of constant curvature

Outline

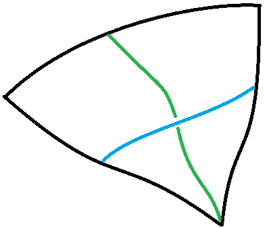
Intrinsic simplices and non-degeneracy

Convex hulls

Natural way to “fill in” a simplex?

Convex hull bad:
-Generally convex hull three points not two dimensional
-Stronger conjectured not closed

Berger 2001 (panoramic overview):
‘It appears as if there is no canonical method to fill up a triangle, or more general simplex, in a generic Riemannian manifold. But this is not true -the problem is solved by the notion of center of mass, modeled on Euclidean geometry.’



Centres of mass in Euclidean space

Weighted average of points

$$\sum \mu_i v_i$$

assume $\sum \mu_i = 1$.
Generalizes to

$$\int p \, \mathrm{d}\mu(p)$$

is where the minimum of

$$P_{\mathbb{R}^n}(x) = \frac{1}{2} \int \|x - p\|^2 \mathrm{d}\mu(p),$$

is attained

Riemannian centres of mass

Centre of mass

$$\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_M(x, v_i)^2$$

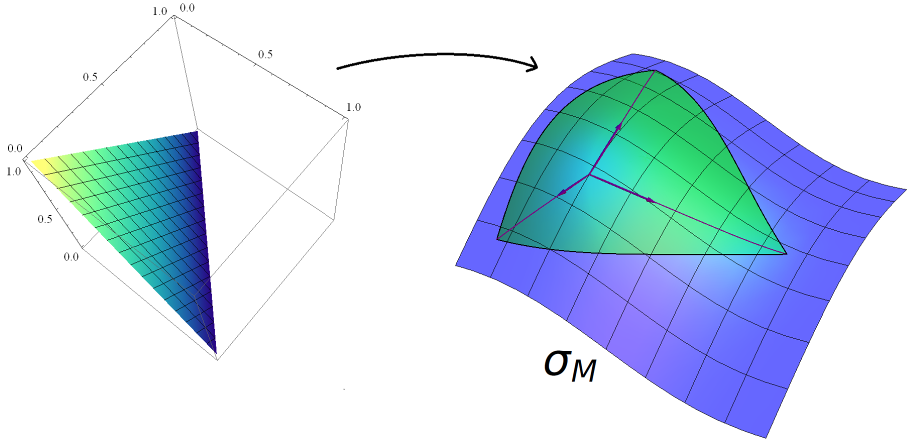
barycentric coordinates: $\lambda_i \geq 0$, $\sum \lambda_i = 1$

$$\begin{aligned} \mathcal{B}_{\sigma^j} : & \boldsymbol{\Delta}^j \to M \\ & \lambda \mapsto \operatorname*{argmin}_{x \in \overline{B}_\rho} \mathcal{E}_\lambda(x) \end{aligned}$$

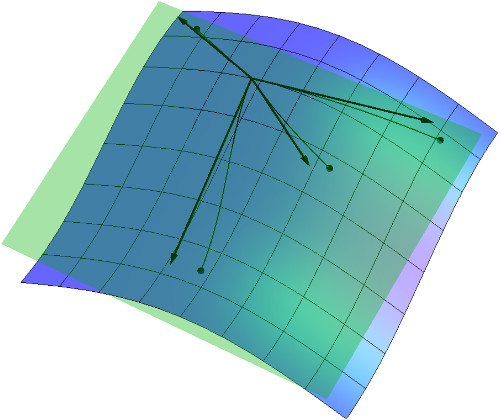
$\boldsymbol{\Delta}^j$ the standard Euclidean j-simplex, $\boldsymbol{\sigma}_M$ image

Point where minimum $\mathcal{E}_\lambda(x)$ is attained is characterized by $\sum \lambda_i \exp_x^{-1}(v_i) = 0$. (generalization of $\sum \lambda_i (v_i - x) = 0$)

Smooth map



Exponential map



Notation

$v_i(x) = \exp_x^{-1}(v_i)$,
$\sigma(x) = \{v_0(x), \ldots, v_j(x)\}$
$\subset T_x M$,
injectivity radius ι_M

$v_i(x) = \exp_x^{-1}(v_i)$ is smooth

Non-degeneracy; a consequence of linear independence

Definition

A Riemannian simplex $\boldsymbol{\sigma}_M$ is *non-degenerate* if $\boldsymbol{\sigma}_M$ is diffeomorphic to the standard simplex $\boldsymbol{\Delta}^n$

Lemma (Consequences of linear independence)

If tangents to geodesics connecting any n (in neighbourhood) to some subset $v_0, \ldots, v_{j-1}, v_{j+1}, \ldots v_n$ (may depend on x) are linearly independent then

- *the map $\boldsymbol{\Delta}^n \to \sigma_M$ is bijective*
- *The inverse of $\boldsymbol{\Delta}^n \to \sigma_M$ is smooth*

$\Delta^n \to \sigma_M$ is bijective

Proof by contradiction

$$\sum \lambda_i \exp_x^{-1}(v_i) = \sum \tilde{\lambda}_i \exp_x^{-1}(v_i) = \sum \tilde{\lambda}_i v_i(x) = 0$$

with $\sum \lambda_i = \sum \tilde{\lambda}_i = 1$. Because $v_0(x), \ldots, \hat{v}_j(x), \ldots, v_n(x)$ linearly independent $\lambda_j \neq 0, \tilde{\lambda}_j \neq 0$. So

$$\frac{\lambda_0}{\lambda_j} v_0(x) + \ldots + \frac{\lambda_{j-1}}{\lambda_j} v_{j-1}(x) + \frac{\lambda_{j+1}}{\lambda_j} v_{j+1}(x) + \ldots + \frac{\lambda_n}{\lambda_j} v_n(x) = \frac{\tilde{\lambda}_0}{\tilde{\lambda}_j} v_0(x) + \ldots + \frac{\tilde{\lambda}_{j-1}}{\tilde{\lambda}_j} v_{j-1}(x) + \frac{\tilde{\lambda}_{j+1}}{\tilde{\lambda}_j} v_{j+1}(x) + \ldots + \frac{\tilde{\lambda}_n}{\tilde{\lambda}_j} v_n(x).$$

Contradiction

Inverse of $\boldsymbol{\Delta}^n \to \sigma_M$ is smooth

Proof

$$\sum \lambda_i \exp_x^{-1}(v_i) = \sum \lambda_i v_i(x) = 0,$$

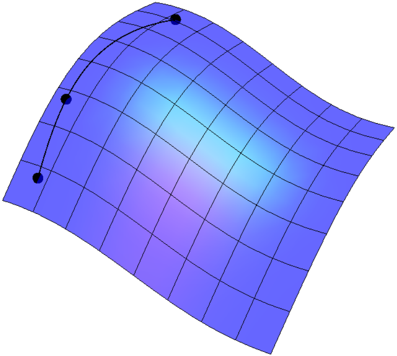
$\lambda_j \neq 0$ because if $\lambda_j = 0$ then

$$\sum_{i \neq j} \lambda_i v_i(x) = 0,$$

contradicting linear independence. So

$$(v_0(x), \ldots, v_{j-1}(x), v_{j+1}(x), \ldots v_n(x))^{-1} v_j(x) = \left(\frac{\lambda_i}{\lambda_j}\right).$$

Linear independence 2D



In two dimensions linear independence easy (Rustamov 2010)

- If $\exp_x^{-1}(v_0) = v_0(x)$, $v_1(x)$, $v_2(x)$ do not span T_xM then they are co-linear.
- Equivalent to v_0, v_1 and v_2 lying on geodesic.

Friedland's bounds

Stability of determinants

$$|\det(A+E)-\det(A)| \leq n \max\{\|A\|_p, \|A+E\|_p\}^{n-1}\|E\|_p$$

with A and E, $n\times n$-matrices
$\|\cdot\|_p$ p-norm on matrices $1\leq p\leq\infty$:

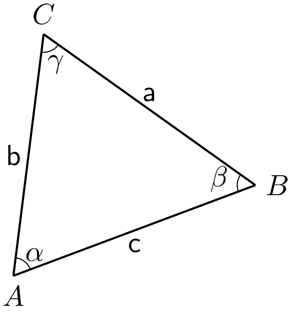
$$\|A\|_p = \max_{x\in\mathbb{R}^n}\frac{|Ax|_p}{|x|_p},$$

p-norm on vectors:

$$|w|_p = \left((w_1)^p + \ldots + (w_n)^p\right)^{1/p}$$

Topogonov comparison theorem

Triangles and cosine rules



Cosine rules:

$$\cos\frac{a}{k} = \cos\frac{b}{k}\cos\frac{c}{k} + \sin\frac{b}{k}\sin\frac{c}{k}\cos\alpha,$$

in a space $\mathbb{H}(1/k^2)$ of sectional curvature $1/k^2$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

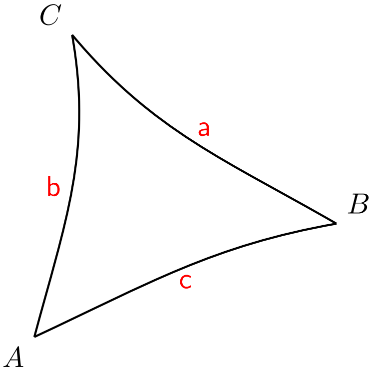
in Euclidean space $\mathbb{R}^n$

$$\cosh\frac{a}{k} = \cosh\frac{b}{k}\cosh\frac{c}{k} - \sinh\frac{b}{k}\sinh\frac{c}{k}\cos\alpha,$$

in a space $\mathbb{H}(-1/k^2)$ of sectional curvature $-1/k^2$

Geodesic triangles

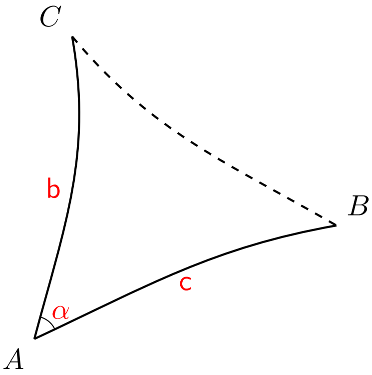
Geodesic triangle T: Three minimizing geodesics connecting three points on a arbitrary manifold (no interior)



Alexandrov triangle: A geodesic triangle with same edge lengths on space of constant curvature $\mathbb{H}(\Lambda_*)$.

Hinges

Hinge: Two minimizing geodesics connecting three points and enclosed angle on a arbitrary manifold



Rauch hinge: A hinge with the same edge lengths and enclosed angle on a space of constant curvature $\mathbb{H}(\Lambda_*)$.

Topogonov comparison theorem

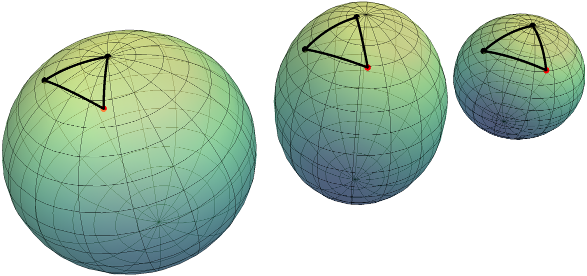
Manifold M, sectional curvatures $\Lambda_- \leq K \leq \Lambda_+$.

Given: Geodesic triangle T on M then exist T_{Λ_-}, T_{Λ_+} on $\mathbb{H}(\Lambda_-)$, $\mathbb{H}(\Lambda_+)$ and

$$\alpha_{\Lambda_-} \leq \alpha \leq \alpha_{\Lambda_+},$$

Given: Hinge on M then exist Rauch hinges on $\mathbb{H}(\Lambda_-)$, $\mathbb{H}(\Lambda_+)$ and the length of the closing geodesics satisfy

$$c_{\Lambda_-} \geq c \geq c_{\Lambda_+}.$$



Non-degeneracy criteria for simplices modeled on simplices in Euclidean space

Setting

Setting:

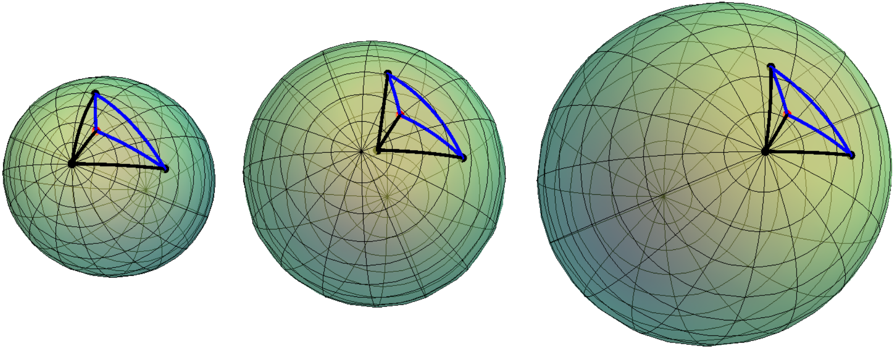
- Manifold M with bounded curvature $|K| \leq \Lambda$.
- Points $\{v_0, \ldots, v_n\}$ in a small ball in M: vertices.
- Choose vertex v_r.
- $\sigma^{\mathbb{E}}(v_r)$ convex hull of the $\exp_{v_r}^{-1}(v_i) = v_i(v_r)$.

Goal:

- Give conditions on $\sigma^{\mathbb{E}}(v_r)$ that imply non-degeneracy

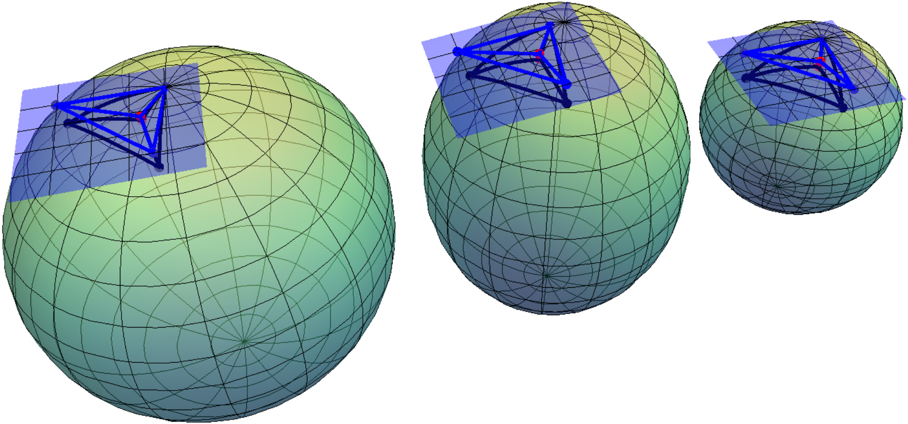
Step 1

We know all the geodesics emanating from v_r. Topogonov for Hinges bounds the length of blue geodesics in the middle by those on the side.



Step 2

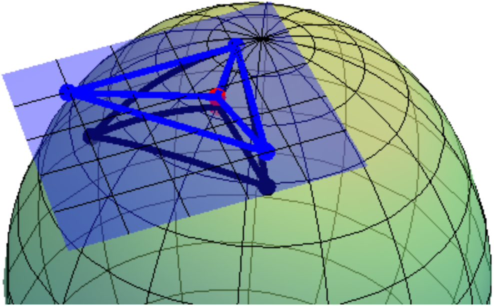
In small neighbourhood the lengths of geodesics on the left and right are close to the lengths in the tangent space (via $\exp^{-1}_{\mathbb{H}(\Lambda_*)}$)

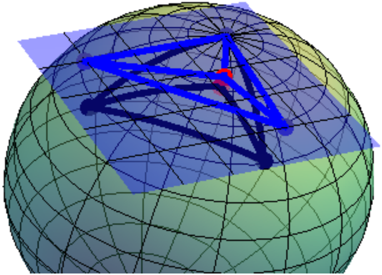


This implies that the same holds for M in the middle.

Step 2

In small neighbourhood the lengths of geodesics on the left and right are close to the lengths in the tangent space (via $\exp^{-1}_{\mathbb{H}(\Lambda_*)}$)

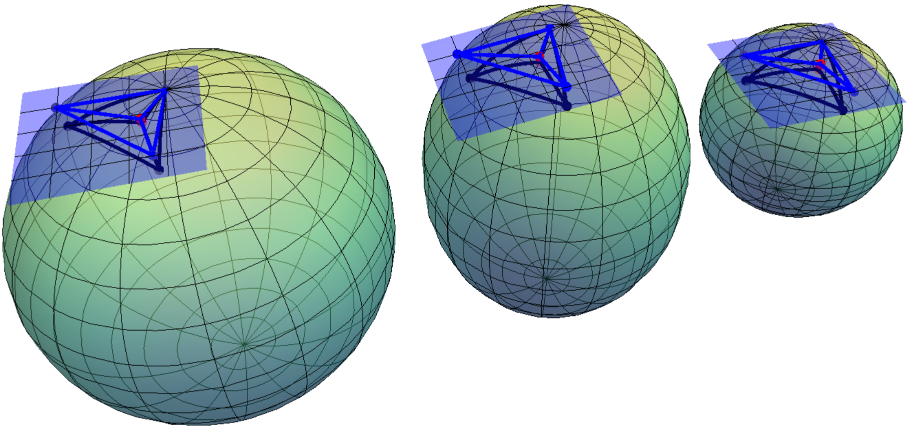




This implies that the same holds for M in the middle.

Step 3

Use the Toponogov comparison theorem for geodesic triangles to conclude that the angles (and inner product) are close to those in the tangent space



Step 4

Gram matrix (in n directions)

$$(\langle \exp_x^{-1} v_i, \exp_x^{-1} v_l \rangle)_{i,l \neq j} = (\langle v_i(x), v_l(x) \rangle)_{i,l \neq j}$$

is close to

$$(\langle v_i(v_r) - x(v_r), v_l(v_r) - x(v_r) \rangle)_{i,l \neq j}$$

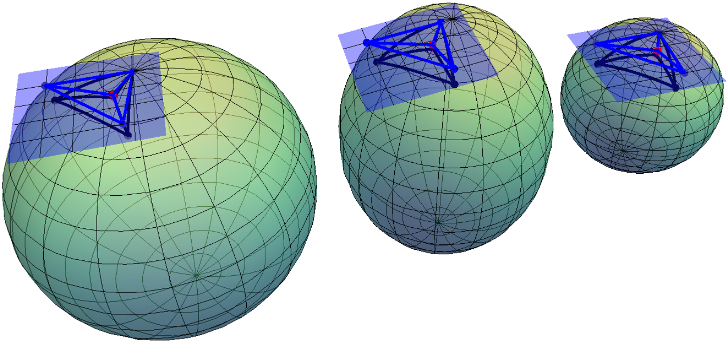
Determinants:

- Friedlands result on stability of determinants gives that determinants are close
- determinants zero iff n tangent vectors linearly independent
- determinants are like volume squared, gives a quality measure

Step 5

- (before) linear independence implies diffeomorphism
- (here) If normalized volume simplex is large enough then linear independent

Gives non-degeneracy criteria



Non-degeneracy criteria for simplices modeled on simplices in spaces of constant curvature

Setting

Setting:

- Manifold M with bounded curvature $\Lambda_- \leq K \leq \Lambda_+$, with $0 < \Lambda_-$ or $\Lambda_+ < 0$.
- Points $\{v_0, \ldots, v_n\}$ in a small ball in M: vertices.
- Choose vertex v_r.
- Model on $\sigma_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(v_r)$ convex hull of $\exp_{\mathbb{H}(\Lambda_{\mathrm{mid}})} \circ \exp_{v_r}^{-1}(v_i) = v_i(v_r)$, with $\Lambda_{\mathrm{mid}} \in [\Lambda_-, \Lambda_+]$.
  If $\Lambda_-, \Lambda_+ > 0$

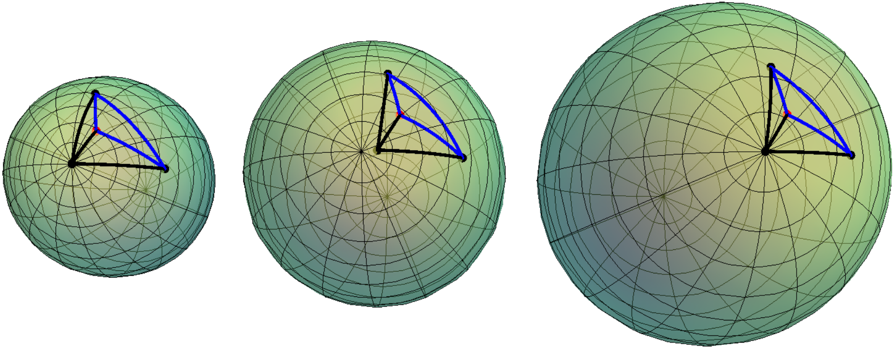
$$\Lambda_{\mathrm{mid}} = \frac{1}{2}(\Lambda_- + \Lambda_+).$$

Goal:

- Give conditions on $\sigma_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(v_r)$ that imply non-degeneracy

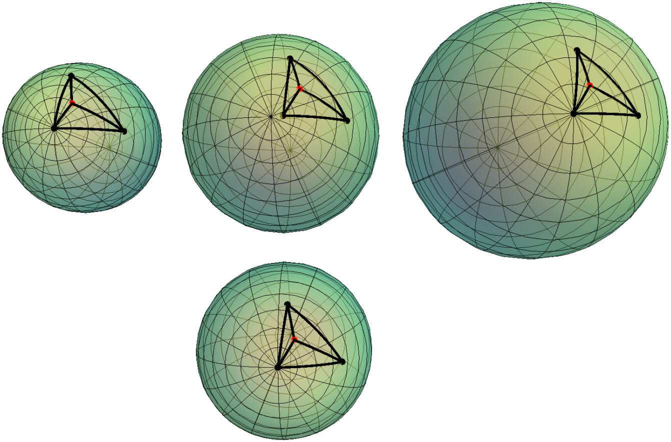
Step 1

We know all the geodesics emanating from v_r. Topogonov for Hinges bounds the length of blue geodesics in the middle by those on the side.



Step 2

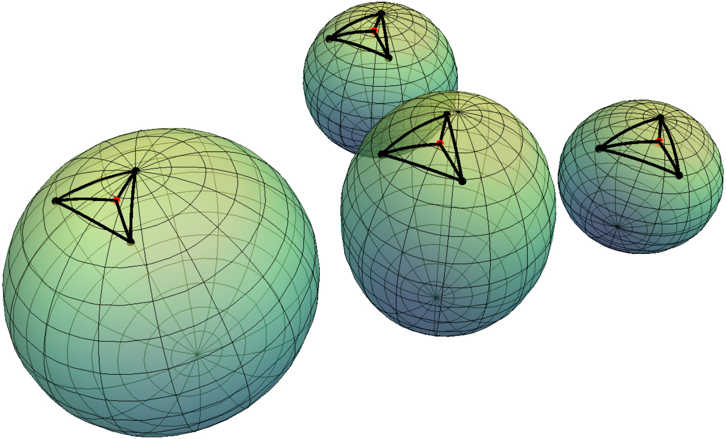
In small neighbourhood the lengths of geodesics on the left and right are close to the lengths on $\mathbb{H}(\Lambda_{\mathrm{mid}})$ (via $\exp_{\mathbb{H}(\Lambda_{\mathrm{mid}})} \circ \exp^{-1}_{v_r, M}$).



This implies that the same holds for M in the middle.

Step 3

Use the Toponogov comparison theorem for geodesic triangles to conclude that the angles (and inner product) are close to those on $\mathbb{H}(\Lambda_{\mathrm{mid}})$



Step 3 (continued)

These inner products are:

$$\frac{1}{\Lambda_{\mathrm{mid}}} \sin\left(\sqrt{\Lambda_{\mathrm{mid}}} d_M(x, v_i)\right) \sin\left(\sqrt{\Lambda_{\mathrm{mid}}} d_M(x, v_j)\right) \cos\theta_{ij,M} \qquad \text{(elliptic)}$$

$$\frac{1}{\Lambda_{\mathrm{mid}}} \sinh\left(\sqrt{\Lambda_{\mathrm{mid}}} d_M(x, v_i)\right) \sinh\left(\sqrt{\Lambda_{\mathrm{mid}}} d_M(x, v_j)\right) \cos\theta_{ij,M} \qquad \text{(hyperbolic)}$$

from the cosine rules:

$$\cos\frac{a}{k} = \cos\frac{b}{k}\cos\frac{c}{k} + \sin\frac{b}{k}\sin\frac{c}{k}\cos\alpha,$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$\cosh\frac{a}{k} = \cosh\frac{b}{k}\cosh\frac{c}{k} - \sinh\frac{b}{k}\sinh\frac{c}{k}\cos\alpha$$

Step 4

Gram matrix for n directions (elliptic case, hyperbolic similar)

$$\left(\frac{1}{\Lambda_{\text{mid}}}\sin(\sqrt{\Lambda_{\text{mid}}}d_M(x,v_i))\sin(\sqrt{\Lambda_{\text{mid}}}d_M(x,v_j))\cos\theta_{ij,M}\right)_{i,l\neq j}$$

is close to

$$\left(\frac{1}{\Lambda_{\text{mid}}}\sin\left(\sqrt{\Lambda_{\text{mid}}}\,d_{\mathbb{H}(\Lambda_{\text{mid}})}(x(v_r),v_i(v_r))\right)\right.$$
$$\left.\sin\left(\sqrt{\Lambda_{\text{mid}}}d_{\mathbb{H}(\Lambda_{\text{mid}})}(x(v_r),v_l(v_r))\right)\cos\theta_{il,\mathbb{H}(\Lambda_{\text{mid}})}\right)_{i,l\neq j}$$

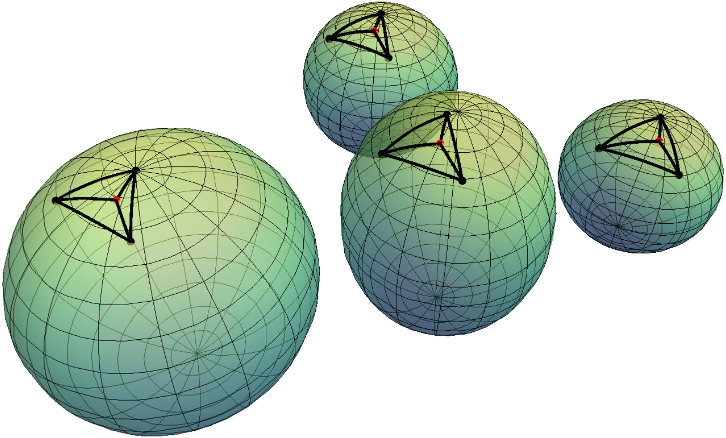
with $x(w) = \exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp_w^{-1} x$
Determinants:

- Friedlands result on stability of determinants
- determinants zero iff n tangent vectors linearly independent
- determinants gives a new quality measure

Step 5

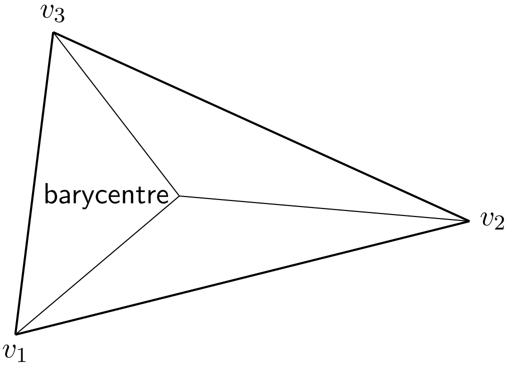
- (before) linear independence implies diffeomorphism
- (here) If quality is large enough then linear independent

Gives non-degeneracy criteria



Quality measures for simplices in spaces of constant curvature

Barycentre and division of volume



The barycentre is the point where the maximum subvolume is minimized:

$$Q_{\mathbb{R}^n}(\sigma) = \min_{y \in \mathbb{R}^n} \max_j \left\{ \det \left((y - v_i) \cdot (y - v_l) \right)_{i,l \neq j} \right\} = \left(\frac{n \cdot \mathrm{volume}(\sigma)}{n+1} \right)^2$$

Quality for positive curvature spaces

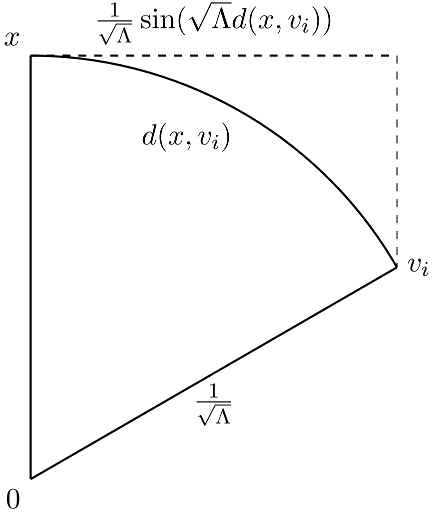
Replacement for inner product

$$\frac{1}{\Lambda}\sin\sqrt{\Lambda}d_M(x,v_i)\sin\sqrt{\Lambda}d_M(x,v_j)\cos\theta_{ij,M}$$

New quality measure

$$\begin{aligned} &Q_{\mathbb{H}(\Lambda)}(\sigma_{\mathbb{H}(\Lambda)}(v_r)) \\ &\quad = \min_{y\in\mathbb{H}(\Lambda)}\max_j \Bigg\{\det\Bigg(\frac{1}{\Lambda}\sin\left(\sqrt{\Lambda}\,d_{\mathbb{H}(\Lambda)}(y,v_i(v_r))\right)\cdot \\ &\qquad\qquad \sin\left(\sqrt{\Lambda}d_{\mathbb{H}(\Lambda)}(y,v_l(v_r))\right)\cos\theta_{il}\Bigg)_{i,l\neq j}\Bigg\}, \end{aligned}$$

Geometric interpretation for positive curvature



For standard embedded sphere

$$\frac{1}{\Lambda} \sin(\sqrt{\Lambda} d(x, v_i)) \sin(\sqrt{\Lambda} d(x, v_j)) \cos \theta_{ij,M}$$

is inner product after projection onto the tangent space.

$Q_{\mathbb{H}(\Lambda)}$ is of the form volume squared in tangent space via projection.

Quality for spaces of negative curvature

$$Q_{\mathbb{H}(\Lambda)} = \min_{x\in N} \max_{j} \Bigg\{ \det \left(\frac{1}{|\Lambda|} \sinh \left(\sqrt{|\Lambda|} d_{\mathbb{H}^n(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(x, v_i(v_r)) \right) \right. \\ \left. \sinh \left(\sqrt{|\Lambda|} d_{\mathbb{H}^n(\Lambda)}(x, v_l(v_r)) \right) \cos\theta_{il} \right)_{i,l\neq j} \Bigg\}$$

Similar geometric interpretation using the hyperboloid model in Minkowski space.

Both inner products have the right Euclidean limit

$$\lim_{\Lambda\to 0} \frac{1}{\Lambda} \sin(\sqrt{\Lambda} b) \sin(\sqrt{\Lambda} c) \cos\theta = bc\cos\theta$$
$$\lim_{\Lambda\to 0} \frac{1}{\Lambda} \sinh(\sqrt{\Lambda} b) \sinh(\sqrt{\Lambda} c) \cos\theta = bc\cos\theta$$

My problems: 1. Quality measures on space form with positive curvature

Is our quality measure the natural one?
Easier ones can be imagined, but must satisfy:

- Small triangles with large volume compared to edge length are good
- Large triangles with vertices near the equator are bad, even if the vertices are evenly spaced

To put it differently, quality of an equilateral triangle should decrease with size.

My problems: 2. Quality measures on space form with negative curvature

Is our quality measure the natural one?

- We use the Minkowski model, is this the most natural way to look at things?
  Expressions bounds (as we shall see) are complicated so maybe not.
- Is there a natural scaling requirement? In the elliptic case the quality decreases with size, in Euclidean it stays the same, does it increase or decrease here?

To put it differently, should quality of an equilateral triangle increase or decrease with size? We do not know.

Results

Non-degeneracy result

Theorem (Non-degeneracy criteria for positive curvature)

Let M manifold with $0 < \Lambda_- \leq K \leq \Lambda_+$. $v_0, \dots, v_n$ vertices on M within a geodesic ball of radius $\frac{1}{2}\tilde{D}$ with centre v_r, where $\tilde{D} \leq 1/(2\sqrt{\Lambda_+})$. Simplex is non-degenerate if

$$\frac{Q_{\mathbb{H}(\Lambda_{mid})}(\sigma_{\mathbb{H}(\Lambda_{mid})}(v_r))}{(2\tilde{D})^{2n}} \geq n \left|\Lambda_- - \Lambda_+\right| \tilde{D}^2$$

$\sigma_{\mathbb{H}(\Lambda_{mid})}(v_r)$ *the simplex on* $\mathbb{H}(\Lambda_{mid})$ *with vertices* $v_i(v_r)$ *defined by* $v_i(v_r) = \exp_{\mathbb{H}(\Lambda_{mid})} \circ \exp^{-1}_{v_r,M}(v_i)$ *and*

$$\Lambda_{mid} = \frac{1}{2}(\Lambda_- + \Lambda_+).$$

Theorem (Non-degeneracy criteria for negative curvature)

Let M be a manifold with $\Lambda_- \leq K \leq \Lambda_+ < 0$. Given vertices $v_0, \dots, v_n$ on M within geodesic ball of radius $\frac{1}{2}\tilde{D}$ with centre v_r. Simplex is non-degenerate if

$$
\begin{aligned}
|\Lambda_{mid}|^n Q_{\mathbb{H}(\Lambda_{mid})}(\sigma_{\mathbb{H}(\Lambda_{mid})}(v_r)) > \; & n(\sinh \sqrt{|\Lambda_-|}\tilde{D})^{2(n-1)} \\
& \cdot \left(2 + 2\cosh\left(\sqrt{|\Lambda_-|}\tilde{D}\right) + |\Lambda_-|^2 \cosh^2\left(\sqrt{|\Lambda_-|}\tilde{D}\right) \frac{11\tilde{D}^4}{4!}\right) \\
& \cdot \left| |\Lambda_-| \cosh^2 \sqrt{|\Lambda_-|}\tilde{D} - |\Lambda_+| \cosh^2 \sqrt{|\Lambda_+|}\tilde{D} \right| |\Lambda_{mid}| \frac{11\tilde{D}^4}{2 \cdot 4!}
\end{aligned}
$$

$\sigma_{\mathbb{H}(\Lambda_{mid})}(v_r)$ *simplex on* $\mathbb{H}(\Lambda_{mid})$*, vertices* $\exp_{\mathbb{H}(\Lambda_{mid})} \circ \exp^{-1}_{v_r,M}(v_i)$ *and*

$$
\begin{aligned}
& \left| |\Lambda_{mid}| \cosh^2 \sqrt{\Lambda_{mid}}\tilde{D} - |\Lambda_-| \cosh^2 \sqrt{|\Lambda_-|}\tilde{D} \right| \\
= & \left| |\Lambda_{mid}| \cosh^2 \sqrt{\Lambda_{mid}}\tilde{D} - |\Lambda_+| \cosh^2 \sqrt{|\Lambda_+|}\tilde{D} \right|.
\end{aligned}
$$

Comment on the quality bound

The quality bound looks awful

$$\left| |\Lambda_-| \cosh^2 \sqrt{|\Lambda_-|} \tilde{D} - |\Lambda_+| \cosh^2 \sqrt{|\Lambda_+|} \tilde{D} \right|$$

but it is roughly speaking proportional to $|\Lambda_- - \Lambda_+|$. Implies that the bounds on the quality go to zero as $|\Lambda_- - \Lambda_+|$ tends to zero.

Questions?